Quantum Background Independence and Witten Geometric Quantization of the Moduli of CY Threefolds.

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Abstract

In this paper we study two different topics. The first topic is the applications of the geometric quantization scheme of Witten introduced in [2] and [16] to the problem of the quantum background independence in string theory. The second topic is the introduction of a $\mathbb{Z}$ structure on the tangent space of the moduli space of polarized CY threefolds $\mathcal{M}(\mathrm{M})$. Based on the existence of a $\mathbb{Z}$ structure on the tangent space of the moduli space of polarized CY threefolds we associate an algebraic integrable structure on the tangent bundle of $\mathcal{M}(\mathrm{M})$. In both cases it is crucial to construct a flat $Sp(2h^{2,1},\mathbb{R})$ connection on the tangent bundle of the moduli space $\mathcal{M}(\mathrm{M})$ of polarized CY threefolds. In this paper we define a Higgs field on the tangent bundle of the moduli space of CY threefolds. Combining this Higgs field with the Levi-Cevita connection of the Weil-Petersson metrics on the moduli space of three dimensional CY manifolds, we construct a new $Sp(2h^{2,1},\mathbb{R})$ connection, following the ideas of Cecotti and Vafa. Using this flat connection, we apply the scheme of geometric quantization introduced by Axelrod, Della Pietra and Witten to the tangent bundle of the moduli space of three dimensional CY manifolds to realize Witten program in [37] of solving the problem of background quantum independence for topological string field theories. By modifying the calculations of E. Witten done on the flat bundle $R^3\pi_*\mathbb{C}$ to the tangent bundle of the moduli space of CY threefolds, we derive the holomorphic anomaly equations of Bershadsky, Cecotti, Ooguri and Vafa as flat projective connection.

Contents

1 Introduction

By definition a Calabi-Yau (CY) manifold is a compact complex $n-$dimensional Kähler manifold M with a holomorphic $n-$form Ω_{M} which has no zeroes and

$H^0\left(\mathrm{M},\Omega^k_{\mathrm{M}}\right) = 0$ for $0 < k < n$. Calabi-Yau manifolds are playing important role in string theory. The powerful ideas from string theory played a very important role in the recent developments in some branches of mathematics and especially in the study of moduli of CY manifolds. In this paper we will study moduli space of CY threefolds based on the ideas introduced in [7], [4] and [37].

In [32] and [30] it was proved that there are no obstructions to the deformations of the complex structures on CY manifolds. This means that the local moduli space of CY manifolds is smooth of dimension

$$h^{n-1,1} = \dim_{\mathbb{C}} H^1\left(\mathrm{M},\Omega^{n-1}_{\mathrm{M}}\right).$$

From the theory of moduli of polarized algebraic manifolds developed by Viehweg in [35] it follows that the moduli space of polarized CY manifolds is a quasi-projective variety.

The moduli space $\mathcal{M}$ of three dimensional CY manifolds has a very rich structure. According to the theory of variations of Hodge structures there exists is a well defined map from the moduli space of polarized CY manifolds to $\mathbb{P}\left(H^n\left(\mathrm{M},\mathbb{Z}\right)\otimes\mathbb{C}\right)$ which is called the period map. It assigns to each point τ of the moduli space the line in $H^n\left(\mathrm{M},\mathbb{Z}\right)\otimes\mathbb{C})$ spanned by the cohomology class represented by the non-zero holomorphic n-form. According to local Torelli Theorem the period map is a local isomorphism. Local Torelli Theorem implies that locally the moduli space of CY manifolds can be embedded in $\mathbb{P}\left(H^n\left(\mathrm{M},\mathbb{Z}\right)\otimes\mathbb{C}\right)$. When the dimension n of the CY manifold is odd, Griffiths and Bryant noticed that the intersection form on $H^n(\mathrm{M},\mathbb{Z})$ defines on the standard charts $U_i = \mathbb{C}^{n-1}$ of $\mathbb{P}\left(H^n\left(\mathrm{M},\mathbb{Z}\right)\otimes\mathbb{C}\right)$ holomorphic one forms α_i such that $d\alpha_i$ is a skew symmetric form of maximal rank on $\mathbb{C}^{n-1}$. This means that on $U_i = \mathbb{C}^{n-1}$ a natural contact structure is defined. In the case of three dimensional CY manifolds Griffiths and Bryant proved that the restrictions of $d\alpha_i$ on the tangent space of the image of the local moduli space of CY manifolds is zero. Thus the image of the local moduli space is a Legandre submanifold. See [3]. Arnold described the local structure of the Legandre submanifolds in a contact manifold in [1]. This description implies the existence of a generating holomorphic function for the local moduli space.

Based on the work [3] A. Strominger noticed that the potential of the Weil-Petersson metric on the local moduli space can be expressed through the generating holomorphic function. See [28]. By using this observation, Strominger introduced the notion of special Kähler geometry. V. Cortes showed that on the tangent bundle of the special Kähler manifold one can introduce a Hyper-Kähler structure. See [8] and [12]. From here it follows that on the tangent bundle of $\mathcal{M}\left(\mathrm{M}\right)$ one can introduce a Hyper-Kähler structure. Earlier R. Donagi and E. Markman constructed in [9] an analytically completely integrable Hamiltonian system which is canonically associated with the family of CY manifolds over the relative dualizing line bundle over the moduli space $\mathcal{M}\left(\mathrm{M}\right)$. They showed that the space of the Griffiths intermediate Jacobians, associated with the family of three dimensional CY manifolds on $\mathcal{M}$ carries a Hyper-Kähler structure. B. Dubrovin introduced the notion of Frobenius manifolds in [10]. The relations

of the structure of Frobenius manifolds and Gromov-Witten invariants were studied by Yu. I. Manin and M. Kontsevich in [23].

The importance of all these structures is justified by the work of Candelas and coauthors in their seminal paper [6]. In this paper Candelas and his coauthors gave an explicit formula for the number of rational curves on the quintic hypersurface in the four dimensional projective space. M. Kontsevich defined the correct compactification of the stable maps and realized that one can use the localization formula for the computations of the rational curves. See [19]. Recently B. Lian, K. Liu and Yau gave a rigorous mathematical proof of the Candelas formula in [20]. See also the important paper of Givental [13].

In this paper we study two different topics. The first topic is the applications of the geometric quantization scheme of Witten introduced in [2] to the problem of the quantum background independence in string theory. The second topic is the introduction of a $\mathbb{Z}$ structure on the tangent space of the moduli space of polarized CY threefolds $\mathcal{M}\,(\mathrm{M})$ and thus we associate an algebraic integrable structure on the tangent bundle of $\mathcal{M}\,(\mathrm{M})$. For both topics it is crucial to construct a flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection on the tangent bundle of the moduli space $\mathcal{M}\,(\mathrm{M})$ of polarized CY threefolds.

The problem of the quantum background independence was addressed in [37]. In the paper [37] Witten wrote:

"Finding the right framework for intrinsic, background independent formulation of string theory is one of the main problems in the subject, and so far has remained out of reach..."

In fact in [37] a program was outlined how one can solve the problem of the background independence in the topological field theory:

"Though the interpretation of the holomorphic anomaly as an obstruction to background independence eliminates some thorny puzzles, it is not satisfactory to simply leave matters as this. Is there some sophisticated sense in which background independence does hold? In thinking about this question, it is natural to examine the all orders generalization of the holomorphic anomaly equation, which in the final equation of their paper [4]) Besrshadsky et. al. write the following form. Let F_g be the genus g free energy. Then

$$\overline{\partial_{i'}}F_g=\overline{C}_{i'j'k'}e^{2K}G^{jj'}G^{kk'}\left(D_jD_kF_{g-1}+\frac{1}{2}\sum_r D_jF_r\cdot D_kF_{g-r}\right). \qquad (1)$$

This equation can be written as a linear equation for

$$Z=\exp\left(\frac{1}{2}\sum_{g=0}^{\infty}\lambda^{2g-2}F_g\right), \qquad (2)$$

namely

$$\left(\overline{\partial_{i'}}-\lambda^2\overline{C}_{i'j'k'}e^{2K}G^{jj'}G^{kk'}D_jD_k\right)Z=0 \qquad (3)$$

This linear equation is called a master equation by Bershadsky et. al.; it is similar to the structure of the heat equations obeyed by theta functions...

It would be nice to interpret (3) as a statement of some sophisticated version of background independence. In thinking about this equation, a natural analogy arises with Chern-Simon gauge theory in $2+1$ dimensions. In this theory, an initial value surface is a Riemann surface Σ. In the Hamiltonian formulation of the theory, one construct a Hilbert space $\mathcal{H}$ upon quantization on $\Sigma.\mathcal{H}$ should be obtained from some physical space $\mathbf{W}$ (a moduli space of flat connections on Σ). Because the underlying Chern-Simon Lagrangian does not depend on the choice of the metric, one would like to construct $\mathcal{H}$ in a natural, background independent way. In practice, however, quantization of $\mathbf{W}$ requires a choice of polarization, and there is no natural way or background independent choice of polarization.

The best that one can do is to pick a complex structure J on Σ, whereupon $\mathbf{W}$ gets a complex structure. Then a Hilbert space $\mathcal{H}_J$ is constructed as a suitable space of holomorphic functions (really sections of a line bundle) over $\mathbf{W}$. We denote such function as $\psi(a^i, t'^a)$ where a^i are complex coordinates on $\mathbf{W}$ and t'^a are coordinates parametrizing the choice of J. Now background independence does not hold in a naive sense; ψ can not be independent of $t^{'i}$ (given that it is to be holomorphic on $\mathbf{W}$ in a complex structure dependent on t'^a). But there is a more sophisticated sense in which background independence can be formulated. See [2] and [16]. The $\mathcal{H}_J$ can be identified with each other (projectively) using a (projectively) flat connection over the space of J's. This connection ∇ is such that a covariant constant wave function should have the following property: as J changes, ψ should change by Bogoliubov transformation, representing the effect of a change in the representation used for the canonical commutation relations. Using parallel transport by ∇ to identify the various $\mathcal{H}'_J$ s are realizations determined by a J-dependent choice of the representation of the canonical commutators. Background independence of $\psi(a^i, t'^a)$ should be interpreted to mean that the quantum state represented by ψ is independent of t'^a, or equivalently that ψ is invariant under parallel transport by ∇. Concretely, this can be written as an equation:

$$\left(\frac{\partial}{\partial t'^a} - \frac{1}{4}\left(\frac{\partial J}{\partial t'^a}\omega^{-1}\right)^{ij}\frac{D}{Da^i}\frac{D}{Da^j}\right)\psi = 0. \tag{4}$$

that is analogous to the heat equation for theta functions..."

In [37] the above program is realized on the space $\mathbf{W} = H^3(\mathrm{M}, \mathbb{R})$. Bershadsky, Cecotti, Ooguri and Vafa work on $H^1(\mathrm{M}, T^{1,0}_{\mathrm{M}})$, i.e. the tangent space to the moduli of CY. The space $\mathbf{W} = H^3(\mathrm{M}, \mathbb{R})$ has a natural symplectic form structure given by the intersection pairing

$$\omega(\alpha, \beta) := \int_{\mathrm{M}} \alpha \wedge \beta.$$

The complex structure on M defines a complex structure on $H^3(\mathrm{M}, \mathbb{R})$. On the vector bundle $R^3\pi_*\mathbb{R}$ over the moduli space with a fibre $\mathbf{W} = H^3(\mathrm{M}, \mathbb{R})$ we have a natural flat $\mathbb{S}p(2h^{2,1}+2, \mathbb{R})$ connection. The tangent bundle to

$\omega_{\mathcal{X}/\mathcal{M}\ (\mathrm{M})}$ is naturally isomorphic to $\pi^*\left(R^3\pi_*\mathbb{R}\right)$. Thus on it we have a natural flat $\mathbb{S}p(2h^{2,1}+2,\mathbb{R})$ connection.

In the present paper the program of Witten is realized for the tangent bundle of the moduli space of polarized CY threefolds. One of the most important ingredient in the realization of the Witten program is the construction of a flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection on the tangent bundle of the moduli space $\mathcal{M}(\mathrm{M})$ of polarized CY threefolds. In the present article we constructed such flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection.

The idea of the construction of the flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection on the tangent bundle of the moduli space $\mathcal{M}(\mathrm{M})$ is to modify the unitary connection of the Weil-Petersson metric on $\mathcal{M}(\mathrm{M})$ with a Higgs field to a $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection and then prove that the $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection is flat. The construction of the Higgs field defined on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ is done by using the cup product $\phi_1\wedge\phi_2\in H^2(\mathrm{M},\wedge^2T^{1,0})$ for $\phi_i\in H^1(\mathrm{M},T^{1,0})$, the identifications of $H^1\left(\mathrm{M},\Omega^2_{\mathrm{M}}\right)$ with $H^1(\mathrm{M},T^{1,0})$, $H^2\left(\mathrm{M},\Omega^1_{\mathrm{M}}\right)$ with $H^2(\mathrm{M},\wedge^2T^{1,0})$ and the identification of $H^1\left(\mathrm{M},\Omega^2_{\mathrm{M}}\right)$ with $H^2\left(\mathrm{M},\Omega^1_{\mathrm{M}}\right)$ by the Poincare duality.

The construction of a flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection on the tangent bundle of $\mathcal{M}(\mathrm{M})$ is related to the important example of a special Hyper-Kähler manifold which occurs in four dimensional gauge theories with $N=2$ supersymmetry: the scalars in the vector multiplet lie in a special Kähler manifolds. The moduli space of such theories was studied by Cecotti and Vafa in [7]. They introduced the tt* equations. One of the observation in this paper is that the analogue of the tt* equations in case of the moduli of polarized CY threefolds is the same as the Yang-Mills equations coupled with Higgs fields that were studied by Hitchin in case of Riemann surfaces in [18] and by C. Simpson in general in [27].

The flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection is crucial to apply the geometric quantization method of Witten to the tangent space of the moduli space $\mathcal{M}(\mathrm{M})$ of polarized CY threefolds to solve the problem of the ground quantum independence in the topological field theory. On the basis of the geometric quantization of the tangent bundle of $\mathcal{M}(\mathrm{M})$ we are able to modify the beautiful computations of E. Witten in [37] to obtain a projective connection on some infinite dimensional Hilbert space bundle. We prove that the holomorphic anomaly equations (1) of Bershadsky, Cecotti, Ooguri and Vafa imply that the free energy obtained from the "counting functions" F_g of curves of genus g on a CY manifold M is a parallel section of a projective flat connection. Our computations are based on the technique developed in [32].

The projective connection constructed in [37] is different from ours since we work on different spaces. The difference appeared in the computation of the formula for $\left(dJ\omega^{-1}\right)$. On the space $\mathbf{W}=H^3(\mathrm{M},\mathbb{R})$ Witten obtained that

$$(dJ)_a^{\overline{b}}=2\sum_{c,d}\overline{C}_{acd}g^{d,\overline{b}}$$

where $\left(g_{a,\overline{b}}\right)$ defines the symplectic structure on $\mathbf{W}$ coming from the cup prod-

uct and C_{acd} is the Yukawa coupling. Our formula on $\mathbf{W} = H^1\left(\mathrm{M}, \Omega^2_{\mathrm{M}}\right)$ is

$$(dJ)_a^{\overline{b}} = \sum_{c,d} \overline{C}_{acd} g^{d,\overline{b}},$$

where $\left(g_{a,\overline{b}}\right)$ is the symplectic form obtained from the restriction of the cup product on $H^1\left(\mathrm{M}, \Omega^2_{\mathrm{M}}\right) \subset H^3(\mathrm{M}, \mathbb{R})$. At the end we obtained exactly the formula (4) suggested by E. Witten.

In [4] two equations are derived. One of them is (1). It gives a recurrent relation between F_g 's. The other equation in [4] is (98). These two equations are marked as (3.6) and (3.8) in [4]. According to [4] the free energy Z satisfy the equation (98). One can notice that there is a difference between our equation and the equation (98) for the free energy Z in [4]. The holomorphic anomaly equation (98) in [4] involves the term F_1 while ours do not.

It was pointed out in [37] that the anomaly equations are the analogue of the heat equations for the classical theta functions. Thus they are of second order. From here one can deduce that if we know the functions F_0 and F_1 that count the rational and elliptic curves on M we will know the functions F_g that count all curves of given genus $g > 1$. It was Welters who first noticed that the heat equation of theta functions can be interpreted as a projective connection. See [36]. Later N. Hitchin used the results of [36] to construct a projectively flat connection on a vector bundle over the Teichmüller space constructed from the symmetric tensors of stable bundle over a Riemann surface. See [18]. For other useful applications of the geometric approach to quantization see [16].

The second problem discussed in this paper is about the existence of $\mathbb{Z}$ structure on the tangent bundle of the moduli space $\mathcal{M}(\mathrm{M})$ of polarized CY threefolds. This problem is suggested by the mirror symmetry conjecture since it suggests that

$$H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

can be "identified" on the mirror side with

$$H^0 \oplus H^2 \oplus H^4 \oplus H^6.$$

Thus since by the mirror conjecture $H^{2,1}$ can be identified with H^2 one should expect some natural $\mathbb{Z}$ structure on $H^{2,1}$ invariant under the flat $\mathbb{S}p(2h^{2,1}, \mathbb{R})$ connection. Thus we need to define at a fixed point of the moduli space $\tau_0 \in \mathcal{M}(\mathrm{M})$ a $\mathbb{Z}$ structure on the tangent space $T_{\tau_0, \mathcal{M}(\mathrm{M})} = H^1(\mathrm{M}_{\tau_0}, \Omega^2_{\mathrm{M}_{\tau_0}})$. One way to obtain a natural $\mathbb{Z}$ structure on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ is the following one. Suppose that there exists a point $\tau_0 \in \mathcal{M}(\mathrm{M})$ such that

$$H^{3,0}(\mathrm{M}_{\tau_0}) \oplus H^{0,3}(\mathrm{M}_{\tau_0}) = \Lambda_0 \otimes \mathbb{C}, \tag{5}$$

where Λ_0 is a rank two sublattice in $H^3(\mathrm{M}_{\tau_0}, \mathbb{Z})$. Once we construct such $\mathbb{Z}$ structure on $T_{\tau_0, \mathcal{M}(\mathrm{M})}$, we can use the parallel transport to define a $\mathbb{Z}$ structure on each tangent space of $\mathcal{M}(\mathrm{M})$. Unfortunately the existence of points that (5)

is satisfied is a very rare phenomenon for CY manifolds. There is a conjecture due to B. Mazur and Y. Andre which states that if the moduli space of CY manifold is a Shimura variety then such points are everywhere dense subset. The moduli space of polarized CY manifolds that are not locally symmetric spaces probably will not contain everywhere dense subset of points that correspond to CY manifolds for which

$$H^{3,0}\left(\mathrm{M}_{\tau_0}\right) \oplus \overline{H^{3,0}\left(\mathrm{M}_{\tau_0}\right)} = \Lambda_0 \otimes \mathbb{C},$$

where Λ_0 is a rank two sublattice in $H^3\left(\mathrm{M}_{\tau_0}, \mathbb{Z}\right)$.

The idea of the introduction of the $\mathbb{Z}$ structure on the tangent space of the moduli space $\mathcal{M}\left(\mathrm{M}\right)$ is to consider the deformation space of $\mathrm{M}\times\overline{\mathrm{M}}$. We relate the local deformation space on $\mathrm{M}\times\overline{\mathrm{M}}$ to the variation of Hodge structure of weight two with $p_g = 1$. Such Variations of Hodge structures for the products $\mathrm{M}\times\overline{\mathrm{M}}$ are defined by the two dimensional real subspace $H^{3,0}\left(\mathrm{M}\right) \oplus \overline{H^{3,0}\left(\mathrm{M}\right)}$ in $H^3\left(\mathrm{M}, \mathbb{R}\right)$ is generated by $\operatorname{Re}\Omega_\tau$ and $\operatorname{Im}\Omega_\tau$ and they are parametrized by the symmetric space

$$\mathbb{SO}_0(2, 2h^{2,1}) / \mathbb{SO}(2) \times \mathbb{SO}(2h^{2,1})$$

where the set of points for which (5) holds is an everywhere dense subset. Thus we are in situation similar to the moduli of algebraic polarized K3 surfaces. For the $(\tau, \overline{\nu})$ in the local moduli space of $\mathrm{M}\times\overline{\mathrm{M}}$ that corresponds to $\mathrm{M}_\tau \times \overline{\mathrm{M}_\nu}$ we construct a Hodge structure of weight two

$$H^{2,0}_{\tau,\nu} \oplus H^{1,1}_{\tau,\nu} \oplus \overline{H^{2,0}_{\tau,\nu}}$$

where $H^{2,0}_{\tau,\nu} \oplus \overline{H^{2,0}_{\tau,\nu}}$ is the two dimensional subspace in $H^3\left(\mathrm{M}, \mathbb{R}\right)$ generated by

$$\operatorname{Re}\left(\Omega_{\tau_1} + \overline{\Omega_{\tau_2}}\right) \text{ and } \operatorname{Im}\left(\Omega_{\tau_2} - \overline{\Omega_{\tau_2}}\right).$$

It is not difficult to show that the points (τ, ν) in the local moduli space of $\mathrm{M}\times\overline{\mathrm{M}}$ such that

$$H^{2,0}_{\tau,\nu} \oplus \overline{H^{2,0}_{\tau,\nu}} = \Lambda_1 \otimes \mathbb{R},$$

where Λ_1 is a rank four sublattice in $H^3\left(\mathrm{M}, \mathbb{Z}\right)$ is an everywhere dense subset. Each point (τ, ν) of this everywhere dense subset defines a natural $\mathbb{Z}$ structure on $H^{1,1}_{\tau,\nu}$. Then by using the flat $\mathbb{S}p(2h^{2,1}, \mathbb{R})$ connection on the tangent space of $\mathcal{M}\left(\mathrm{M}\right)$ we define a $\mathbb{Z}$ structure on $H^{2,0}_{\tau,\tau}$ and thus on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. By using this $\mathbb{Z}$ structure we introduce an algebraic integrable structure on the tangent bundle of $\mathcal{M}\left(\mathrm{M}\right)$. In [9] the authors introduced algebraic integrable structure on the tangent bundle of the relative dualizing line bundle of $\omega_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ of the moduli space $\mathcal{M}\left(\mathrm{M}\right)$ of polarized CY threefolds.

All the results in the **Sections 3, 4, 5, 6** and **7** are new. Next we will describe the ideas and the content of each section.

In **Section 2** we review the results of [32] and [30]. The Teichmüller space of the CY manifolds is constructed too.

In **Section 3** we show that the analogue of the tt* equations on $\mathcal{M}\left(\mathrm{M}\right)$ are the same self dual equations that were studied by N. Hitchin and C. Simpson's

in [17] and [26]. Thus tt* equations define a flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection on the bundle $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. On the other hand we know that the tangent bundle $\mathcal{T}_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ is isomorphic to $\mathfrak{L}^*\otimes\ R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$, where $\mathfrak{L}$ is isomorphic to $\pi_*\Omega^3_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. We constructed by using the theory of determinant bundles a holomorphic non-vanishing section $\eta_\tau \in \Gamma\left(\mathcal{M}(\mathrm{M}),(\mathfrak{L})\right)$ in [5]. Thus η_τ defines a flat structure on the tangent bundle $\mathcal{T}_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ of the moduli space of three dimensional CY manifolds $\mathcal{M}\left(\mathrm{M}\right)$. Using the flat structure defined by tt* equations on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ and the flat structure defined by the section η_τ on $\mathfrak{L}$, we define a flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection on the tangent bundle $\mathcal{T}_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ of the moduli space $\mathcal{M}\left(\mathrm{M}\right)$ of three dimensional CY manifolds. We will call this connection the Cecotti-Hitchin-Simpson-Vafa connection and will refer to it as the CHSV connection.

A beautiful theorem of Simpson proved in [26] shows when a quasi-projective variety is covered by a symmetric domain. One can show that the tt* equations can be interpreted in the same way. This will be done in [31].

In **Section 3** we interpreted the holomorphic connection which is defined by the Frobenius Algebra structure on the bundle $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ as a Higgs field. It seems that the paper by Deligne is suggesting that the Higgs field that we constructed is related to variation of mixed Hodge Structure of CY threefolds, when there exists a maximal unipotent element in the mapping class group. See [11].

In **Section 4** we review some basic constructions from [2].

In **Section 5** we used the ideas from [2] and some modifications of the beautiful computations done by E. Witten in [37] to quantize the tangent bundle of $\mathcal{M}\left(\mathrm{M}\right)$. This can be done since we can identify the tangent spaces at each point of the moduli space of CY manifolds by using the parallel transport defined by the flat connection $\mathbb{S}p(2h^{2,1},\mathbb{R})$ defined by Cecotti-Hitchin-Simpson-Vafa and the existence of the non-zero section η_τ of the relative dualizing line bundle over $\mathcal{M}\left(\mathrm{M}\right)$. We will show that the symplectic structure defined by the imaginary part of the Weil-Petersson metric is parallel with respect to the CHSV connection. In this section we construct a projective flat connection on some Hilbert vector bundle associated with the tangent bundle on the moduli space $\mathcal{M}\left(\mathrm{M}\right)$. Based on these results, the method from [37] and the technique developed in [32], we show that holomorphic anomaly equations (1) of Bershadsky, Cecotti, Ooguri and Vafa of the genus $g \geq 2$ imply that the free energy Z defined by (97) is a parallel with respect to a flat projective connection constructed in **Section 6**.

In **Section 6** we will introduce a natural $\mathbb{Z}$ structure on the tangent space of each point of the moduli space of CY threefolds by using the flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection constructed in **Section 3**. In order to do that we introduce the notion of the extended period space of CY threefolds which is similar to the period domain of marked algebraic polarized K3 surfaces. We know from the moduli theory of algebraic polarized K3 surfaces that the points that define K3 surfaces with CM structure form an everywhere dense subset. This follows from the fact that the period domain is an open set on a quadric defined over

$\mathbb{Q}$ in the projective space $\mathbb{P}(\mathbb{Z}^{20} \otimes \mathbb{C})$. This fact together with the existence of a flat $\mathbb{S}p(4h^{2,1}, \mathbb{R})$ connection on the extended period domain will define in a natural way a lattice of maximal rank in the tangent space at each point of the moduli space of CY threefolds. Using the existence of the $\mathbb{S}p(2h^{2,1}, \mathbb{R})$ connection defined by the tt* equation on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$, we define by the parallel translations a $\mathbb{Z}$ structure on the fibres of the bundle $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ at each point of the moduli space of M.

The mirror symmetry suggests that we can identify the second cohomology group of the mirror CY M' with $H^{2,1}$ of the original CY manifold. Since the second cohomology group of a CY manifold has a natural $\mathbb{Z}$ structure, then $H^{2,1}$ of the original CY manifold should also carry a natural $\mathbb{Z}$ structure. This construction suggests that the existence of the natural $\mathbb{Z}$ structure on $H^{2,1}$ is equivalent to the tt* equations.

In **Section 7** we obtain an algebraic integrable system in the sense of R. Donagi and E. Markman using the flat Cecotti-Hitchin-Simpson-Vafa connection. From that we obtain a map from the moduli space of CY manifold M to the moduli space of principally polarized abelian varieties and the CHSV connection is the pull back of the connection defined by R. Donagi and E. Markman on the moduli space of principally polarized abelian varieties. See [9]. We also construct a Hyper-Kähler structure on the tangent bundle of the moduli space $\mathcal{M}(\mathrm{M})$ of polarized CY threefolds. D. Freed constructed Hyper-Kähler structure on the tangent bundle of the relative dualizing sheaf of the moduli space $\mathcal{M}(\mathrm{M})$ of polarized CY threefolds in [12].

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2 Deformation Theory for CY manifolds.

2.1 Review of [32]

Definition 1 *Let M be an even dimensional C^∞ manifold. We will say that M has an almost complex structure if there exists a section $I \in C^\infty(M, Hom(T^*, T^*))$ such that $I^2 = -id$. T is the tangent bundle and T^* is the cotangent bundle on M.*

This definition is equivalent to the following one:

Definition 2 *Let M be an even dimensional C^∞ manifold. Suppose that there exists a global splitting of the complexified cotangent bundle $T^* \otimes \mathbf{C} = \Omega^{1,0} \oplus \Omega^{0,1}$, where $\Omega^{0,1} = \overline{\Omega^{1,0}}$. Then we will say that M has an almost complex structure.*

We are going to define the almost integrable complex structure.

Definition 3 *We will say that an almost complex structure is an integrable one if for each point $x \in M$ there exists an open set $U \subset M$ such that we can find local coordinates $z^1, .., z^n$ such that $dz^1, .., dz^n$ are linearly independent in each point $m \in U$ and they generate $\Omega^{1,0}|_U$.*

It is easy to see that any complex manifold has an almost integrable complex structure.

Definition 4 *Let M be a complex manifold. $\phi \in \Gamma(M, Hom(\Omega^{1,0}, \Omega^{0,1}))$ is called a Beltrami differential.*

Since $\Gamma(\mathrm{M}, Hom(\Omega^{1,0}, \Omega^{0,1})) \backsimeq \Gamma(\mathrm{M}, T^{1,0} \otimes \Omega^{0,1})$, we deduce that locally ϕ can be written as follows:

$$\phi|_U = \sum \phi^{\beta}_{\overline{\alpha}} \overline{dz}^{\alpha} \otimes \frac{\partial}{\partial z^{\beta}}.$$

From now on we will denote by

$$A_{\phi} = \begin{pmatrix} id & \phi(\tau) \\ \overline{\phi(\tau)} & id \end{pmatrix} : T^* \otimes \mathbb{C} \to T^* \otimes \mathbb{C}.$$

We will consider only those Beltrami differentials ϕ such that $\det(A_{\phi}) \neq 0$.

Definition 5 *It is easy to see that the Beltrami differential ϕ defines a new almost complex structure operator $I_{\phi} = A_{\phi}^{-1} \circ I \circ A_{\phi}$.*

With respect to this new almost complex structure the space $\Omega^{1,0}_{\phi}$ is defined as follows; if $dz^1, .., dz^n$ generate $\Omega^{1,0}|_U$, then

$$dz^1 + \phi(dz^1), .., dz^n + \phi(dz^n)$$

generate $\Omega^{1,0}_{\phi}|_U$ and, moreover we have: $\overline{\Omega^{1,0}_{\phi}} \cap \Omega^{1,0}_{\phi} = 0$. The Beltrami differential ϕ defines an integrable complex structure on M if and only if the following equation holds:

$$\overline{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0.$$

where

$$[\phi, \phi]|_U := \sum_{\nu=1}^{n} \sum_{1 \leqq \alpha, \beta \leqq n} \left(\sum_{\mu=1}^{n} \left(\phi^{\mu}_{\overline{\alpha}} \left(\partial_{\mu} \phi^{\nu}_{\overline{\beta}} \right) - \phi^{\mu}_{\overline{\beta}} \left(\partial_{\mu} \phi^{\nu}_{\overline{\alpha}} \right) \right) \right) \overline{dz}^{\alpha} \wedge \overline{dz}^{\beta} \otimes \frac{\partial}{dz^{\nu}}.$$

(See [24].)

The main results in [32] are the two theorems stated bellow:

Theorem 6 *Let M be a CY manifold and let $\{\phi_i\}$ be harmonic (with respect to the CY metric g) representative of the basis in $\mathbb{H}^1(M, T^{1,0})$, then the equation: $\overline{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0$ has a solution in the form:*

$$\phi(\tau_1, .., \tau_N) = \sum_{i=1}^{N} \phi_i \tau^i + \sum_{|I_N| \geqq 2} \phi_{I_N} \tau^{I_N}$$

where $I_N = (i_1, .., i_N)$ is a multi-index,

$$\phi_{I_N} \in C^\infty(M, \Omega^{0,1} \otimes T^{1,0}),$$

$\tau^{I_N} = (\tau^i)^{i_1} ... (\tau^N)^{i_N}$ and there exists $\varepsilon > 0$ such that

$$\phi(\tau) \in C^\infty(M, \Omega^{0,1} \otimes T^{1,0})$$

for $|\tau^i| < \varepsilon$ for $i = 1, .., N$. See [32].

Theorem 7 *Let Ω_0 be a holomorphic n-form on the n dimensional CY manifold M. Let $\{U_i\}$be a covering of M and let $\{z_1^i, .., z_n^i\}$ be local coordinates in U_i such that $\Omega_0 |_{U_i} = dz_1^i \wedge ... \wedge dz_n^i$. Then for each $\tau = (\tau^1, .., \tau^N)$ such that $|\tau_i| < \varepsilon$ the forms on M defined as:*

$$\Omega_t |_{U_i} := (dz_1^i + \phi(\tau)(dz_1^i)) \wedge .. \wedge (dz_n^i + \phi(\tau)(dz_n^i))$$

are globally defined complex n forms Ω_τ on M and, moreover, Ω_τ are closed holomorphic n forms with respect to the complex structure on M defined by $\phi(\tau)$.

Corollary 8 *We have the following Taylor expansion for*

$$\Omega_\tau |_U = \Omega_0 + \sum_{k=1}^{n} (-1)^{\frac{k(k-1)}{2}} \left(\wedge^k \phi\right) \lrcorner \Omega_0. \tag{6}$$

(See [32].)

From here we deduce the following Taylor expansion for the cohomology class $[\Omega_\tau] \in H^n(\mathrm{M}, \mathbf{C})$:

Corollary 9

$$[\Omega_\tau] = [\Omega_0] - \sum_{i=1}^{N} [(\phi_i \lrcorner \Omega_0)] \tau^i + \frac{1}{2} \sum_{i,j=1}^{N} [((\phi_i \wedge \phi_j) \lrcorner \Omega_0)] \tau^i \tau^j + O(\tau^3) \tag{7}$$

(See [32].)

We are going to define the Kuranishi family for CY manifolds of any dimension.

Definition 10 *Let $\mathcal{K} \subset\mathbb{C}^N$ be the polydisk defined by $|\tau^i| < \varepsilon$ for every $i = 1,..,N$, where ε is chosen such that for every $\tau \in \mathcal{K}$, $\phi(\tau) \in C^\infty(M,\Omega^{0,1}\otimes T^{1,0})$, where $\phi(\tau)$ is defined as in Definition 4. On the trivial C^∞ family $M\times\mathcal{K}$ we will define for each $\tau \in \mathcal{K}$ an integrable complex structure $I_{\phi(\tau)}$ on the fibre over τ of the family $M\times\mathcal{K}$,where $I_{\phi(\tau)}$ was defined in Definition 5. Thus we will obtain a complex analytic family $\pi : \mathcal{X} \to \mathcal{K}$ of CY manifolds. We will call this family the Kuranishi family. Thus we introduce also a coordinate system in $\mathcal{K}$. We call this coordinate system a flat coordinate system.*

2.2 Construction of the Teichmüller Space of CY Manifolds

Definition 11 *We will define the Teichmüller space $\mathcal{T}(M)$ of M as follows:*

$$\mathcal{T}(M) := \{\textit{all integrable complex structures on } M\} / \mathbf{Diff}_0(M) ,$$

where $\mathbf{Diff}_0(M)$ is the group of diffeomorphisms of M isotopic to identity.

$\mathbf{Diff}_0(\mathrm{M})$ acts on the complex structures as follows: let $\psi \in \mathbf{Diff}_0(\mathrm{M})$ and let

$$I \in C^\infty(Hom(T^*(\mathrm{M}), T^*(\mathrm{M})),$$

such that $I^2 = -id$, then clearly $\psi^*(I)$ is such that $(\psi^*(I))^2 = -id$. Moreover, if I is an integrable complex structure, then $\psi^*(I)$ is integrable too.

We will call a pair (M, $\{\gamma_1, ..., \gamma_{b_n}\}$) a marked CY manifold, if M is a Calabi-Yau manifold and $\{\gamma_1, .., \gamma_{b_n}\}$ is a basis in $H_n(M, \mathbf{Z})/Tor$. Over the Kuranishi space we have a universal family of marked Calabi-Yau manifolds $\mathcal{X} \to \mathcal{K}$ defined up to an action of a group that acts trivially on the middle homology and preserves the polarizations class. And, moreover, as a C^∞ manifold $\mathcal{X}$ is diffeomorphic to $\mathcal{K}\times$M.

Theorem 12 *The Teichmüller space $\mathcal{T}(M)$ of a Calabi Yau manifold M exists as a complex manifold of dimension $h^{2,1}$.*

Proof: For the proof of Theorem 12 see [22]. ■

2.3 Construction of the Moduli Space

Definition 13 *We will define the mapping class group $\Gamma'(M)$ as follows:*

$$\Gamma'(M) := \mathbf{Diff}^+(M) / \mathbf{Diff}_0(M) ,$$

where $\mathbf{Diff}^+(M)$ is the group of diffeomorphisms preserving the orientation of M and $\mathbf{Diff}_0(M)$ is the group of diffeomorphisms isotopic to identity.

D. Sullivan proved that the mapping class group of any C^∞ manifold of dimension greater or equal to 5 is an arithmetic group. (See [29].) It is easy to

prove that the mapping class group $\Gamma'(\mathrm{M})$ acts discretely on the Teichmüller space $\mathcal{T}(\mathrm{M})$ of the CY manifold M.

We will consider from now on polarized CY manifolds, i.e. a pair $(M, \omega(1,1))$, where

$$[\omega(1,1)] \in H^2(\mathrm{M}, \mathbb{Z}) \cap H^{1,1}(\mathrm{M}, \mathbb{R})$$

is a fixed class of cohomology and it corresponds to the imaginary part of a CY metric. We will define $\Gamma(\mathrm{M})$ as follows:

$$\Gamma_{\omega(1,1)}(\mathrm{M}) := \{\phi \in \Gamma'(\mathrm{M}) \,|\phi([\omega(1,1)]) = [\omega(1,1)]\,\}.$$

From now on we will work with this family.

Theorem 14 *There exists a subgroup $\Gamma(M)$ in $\Gamma_{\omega(1,1)}$ of finite index such that Γ acts without fixed points on the Teichmüller space $\mathcal{T}(M)$. The moduli space $\mathfrak{M}(M) = \mathcal{T}(M)/\Gamma(M)$ is a smooth quasi-projective variety. There exists a family of polarized CY manifolds $\mathcal{Y}(M) \to \mathcal{M}(M) = \mathcal{T}(M)/\Gamma(M)$. The relative dualizing sheaf $\omega_{\mathcal{Y}/\mathcal{M}(M)}$ is a trivial line bundle.*

Proof: Viehweg proved in [35] that the moduli space $\mathcal{M}(\mathrm{M})$ is a quasi projective variety. In [22] it was proved that we can find a subgroup $\Gamma(\mathrm{M})$ in $\Gamma_{\omega(1,1)}(\mathrm{M})$ such that the space $\mathcal{T}(\mathrm{M})/\Gamma(\mathrm{M})$ is a smooth complex manifold. We also proved that over $\mathcal{T}(\mathrm{M})/\Gamma(\mathrm{M}) = \mathcal{M}(\mathrm{M})$ there exists a family of CY manifolds $\mathcal{Y}(\mathrm{M}) \to \mathcal{M}(\mathrm{M})$. In [5] we proved the following Theorem:

Theorem 15 *Let $\mathcal{M}(M) = \mathcal{T}(M)/\Gamma(M)$. Then there exists a global non vanishing holomorphic section η_τ of the line bundle $\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}$ whose L^2 norm $\|\eta_\tau\|^2_{\mathbf{L}^2}$ is equal to $\left(\det_{(0,1)}\right)$, where $\det_{(0,1)}$ is the regularized determinant of the Laplacian of a CY acting on $\Omega^{0,1}_M$ of the CY metric with imaginary class equal to the polarization class and $\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}$ is a trivial holomorphic line bundle.*

Theorem 14 follows from Theorem 15. ■

Corollary 16 *η_τ defines a flat structure on $\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}$.*

2.4 Weil-Petersson Geometry

In our paper [32] we define a metric on the Kuranishi space $\mathcal{K}$ and called this metric, the Weil-Petersson metric. We will review the basic properties of the Weil-Petersson metric which were established in [32]. In [32] we proved the following theorem:

Theorem 17 *Let M be a CY manifold of dimension n and let Ω_M be a non zero holomorphic n form on M such that*

$$(-1)^{\frac{n(n-1)}{2}} \left(\sqrt{-1}\right)^n \int_M \Omega_M \wedge \overline{\Omega_M} = 1.$$

Let g be a Ricci flat (CY) metric on M. Then the map:

$$\psi \in L^2(M, \Omega_M^{0,k} \wedge^m T_M^{1,0}) \to \psi \lrcorner \Omega_M \in L^2(M, \Omega_M^{n-m,k})$$

gives an isomorphism between Hilbert spaces and this map preserves the Hodge decomposition.[32].

Corollary 18 *We can identify the tangent space $T_\tau = H^1(M_\tau, T_\tau^{1,0})$ at each point $\tau \in \mathcal{T}(M)$ with $H^1(M_\tau, \Omega_\tau^{n-1})$, by using the map $\psi \to \psi \lrcorner \Omega_M$.*

Notation 19 *We will denote by*

$$\langle \omega_1, \omega_2 \rangle := \int\limits_{M_\tau} \omega_1 \wedge \overline{\omega_2}. \tag{8}$$

Definition 20 *Let $\psi_1, \psi_2 \in T_\tau = \mathbf{H}^1\left(M_\tau, T_{M_\tau}^{1,0}\right)$ (the space of harmonic forms with respect to the CY metric g.). We will define the Weil-Petersson metric as follows:*

$$\langle \psi_1, \psi_2 \rangle_{WP} := \sqrt{-1} \int\limits_{M_\tau} (\psi_1 \lrcorner \Omega_\tau) \wedge \left(\overline{\psi_2 \lrcorner \Omega_\tau}\right) = \sqrt{-1} \left\langle \psi_1 \lrcorner \Omega_\tau, \overline{\psi_2 \lrcorner \Omega_\tau} \right\rangle$$

and $\|\Omega_\tau\|^2 = 1$. Thus $\langle \psi, \psi \rangle_{WP} > 0$.

The Weil-Petersson metric is a Kähler metric on the Teichmüller space $\mathcal{T}$(M). It defines a natural connection, namely the Levi-Civita connection $\not\!\!/$. We will denote the covariant derivatives in direction $\frac{\partial}{\partial \tau^i}$ at the tangent space of a point $\tau \in \mathcal{T}$(M) defined by ϕ_i by ∇_i. In [32] we proved the following theorem:

Theorem 21 *In the flat coordinate system introduced in Definition 10 the following formulas hold for the curvature operator:*

$$R_{i\bar{j},k\bar{l}} = \delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{j}} - \sqrt{-1}\int\limits_M ((\phi_i \wedge \phi_k)\lrcorner\Omega_M)) \wedge (\overline{(\phi_j \wedge \phi_l)\lrcorner\Omega_M})$$

$$= \delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{j}} - \sqrt{-1}\left\langle (\phi_i \wedge \phi_k)\lrcorner\Omega_M)), (\phi_j \wedge \phi_l)\lrcorner\Omega_M \right\rangle. \tag{9}$$

3 Flat $\mathbb{S}\mathbf{p}(2\mathbf{h}^{2,1}, \mathbb{R})$ Structure on the Moduli Space of CY Threefolds

3.1 A Flat Structure on the Line Bundle $\omega_{\mathcal{Y}(\mathbf{M})/\mathcal{M}(\mathbf{M})}$

The flat structure on the line bundle $\omega_{\mathcal{Y}\text{ (M) }/\mathcal{M}\text{(M)}}$ is defined by Corollary 16.

3.2 Gauss-Manin Connection

Definition 22 *On the Teichmüller space $\mathcal{T}(M)$ we have a trivial bundle namely*

$$\mathcal{H}^n = H^n(M,\mathbb{C}) \times \mathcal{T}(M) \to \mathcal{T}(M).$$

Theorem 14 implies that we constructed the moduli space $\mathcal{M}(M)$ as $\mathcal{T}(M)/\Gamma(M)$. Thus we obtain a natural representation of the group $\Gamma(M)$ into $H^n(M,\mathbb{C})$ and a flat connection on the flat bundle

$$\mathcal{H}^n/\Gamma(M) \to \mathcal{T}(M)/\Gamma(M) = \mathcal{M}(M).$$

This connection is called the Gauss-Manin connection. The covariant derivative in direction ϕ_i of the tangent space $T_{\tau,\mathcal{M}(M)}$ with respect to the Gauss-Manin connection will be denoted by $\mathcal{D}_i$.

The Gauss-Manin connection $\mathcal{D}$ is defined in a much more general situation and it is defined on the moduli space of CY manifolds of dimension $n \geq 3$. We will state explicit formulas for the covariant differentiation $\mathcal{D}_i$ defined by the Gauss-Manin connection. We will fix a holomorphic three form Ω_0 such that

$$-\sqrt{-1}\langle\Omega_0,\Omega_0\rangle = -\sqrt{-1}\int_{\mathrm{M}}\Omega_0 \wedge \overline{\Omega_0} = \|\Omega_0\|^2 = 1.$$

Using the form Ω_0 and theorem 17, we can identify the cohomology groups $H^1\left(\mathrm{M}, T^{1,0}_{\mathrm{M}}\right)$ and $H^1\left(\mathrm{M},\Omega^2_{\mathrm{M}}\right)$ on M:

Proposition 23 *The map*

$$\mathfrak{i} : \psi \to \psi \lrcorner \Omega_M \tag{10}$$

is an isomorphism between the groups $H^1\left(M, T^{1,0}_M\right)$ and $H^1\left(M,\Omega^2_M\right)$.

Proof: Our proposition follows directly from Theorem 17. ■

Remark 24 *From now on in the map (10) we will use for Ω_M the restriction of the holomorphic form η_τ on M defined by Theorem 15.*

Remark 25 *Suppose that M is a three dimensional CY manifold. Then the Poincare map identifies $H^1\left(M,\Omega^2_M\right)$ with $H^2\left(M,\Omega^1_M\right)$. This identification will be denoted by Π, i.e. $\Pi : H^2\left(M,\Omega^1_M\right) \to H^1\left(M,\Omega^2_M\right)$ and it is defined by identifying some basis $\{\Omega_i\}$ of $H^2\left(M,\Omega^1_M\right)$ with the basis $\left\{\overline{\Omega_i}\right\}$ of $H^1\left(M,\Omega^2_M\right)$. So:*

$$\Pi(\Omega_i) := \overline{\Omega_i}. \tag{11}$$

Notation 26 *Using Proposition 23 and Remark 25 one can identify the spaces $H^1\left(M, T^{1,0}_M\right)$ and $H^2\left(M,\Omega^1_M\right)$ for three dimensional CY by using the map F, where*

$$F(\phi) := \Pi(\iota(\phi)) \tag{12}$$

for $\phi \in H^1(M, T^{1,0}_M)$.

Lemma 27 *Let* $\mathfrak{i}^{-1} : H^1\left(M, \Omega^2_M\right) \overset{\llcorner\Omega^*_M}{\simeq} H^1\left(M, T^{1,0}_M\right)$ *be the inverse identification defined by (10). Then*

$$\mathcal{D}_i(\mathfrak{i}(\phi)) = \iota\left(\phi_i\right)\lrcorner\phi \in H^2\left(M_0, \Omega^1_{M_0}\right). \tag{13}$$

Proof: The proof of the lemma follows directly from formula(7). Indeed since $\{\phi_i\}$ is a basis of $H^1(\mathrm{M}, T^{1,0}_{\mathrm{M}})$, then $\{\phi_i\lrcorner\Omega_{\mathrm{M}}\}$ will be a basis of $H^1\left(\mathrm{M}, \Omega^2_{\mathrm{M}}\right)$. In the flat coordinates $(\tau^1, ..., \tau^N)$ introduced in Definition 10 and according to (7) we have

$$[\Omega_\tau] = [\Omega_{\mathrm{M}}] - \sum_{i=1}^{N} \left[(\phi_i\lrcorner\Omega_{\mathrm{M}})\right]\tau^i + \sum_{i,j=1}^{N} \left[(\phi_i \wedge \phi_j)\lrcorner\Omega_{\mathrm{M}}\right]\tau^i\tau^j + O(\tau^3).$$

From Definition 22 and the expression of $[\Omega_\tau]$ given by the above formula, we deduce formula (13). Lemma 22 is proved. ■

3.3 Higgs Fields and the Tangent Space of $\mathcal{M}\left(\mathrm{M}\right)$

Let $\omega_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ be the relative dualizing sheaf on $\mathcal{M}\left(\mathrm{M}\right)$.

Definition 28 *We define the Higgs field of a holomorphic bundle* $\mathcal{E}$ *over a complex manifold* M *as a globally defined holomorphic map* $\Phi : \mathcal{E} \to \mathcal{E}\otimes\Omega^1_M$ *such that*

$$\Phi \circ \Phi = 0. \tag{14}$$

Lemma 29 *The tangent bundle* $T_{\mathcal{T}(M)}$ *of the Teichmüller space* $\mathcal{T}\left(M\right)$ *of any CY manifold* M *is canonically isomorphic to the bundles*

$$T_{\mathcal{T}(M)} \approx Hom\left(\pi_*\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}, R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}\right) \approx$$

$$\left(\pi_*\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}\right)^* \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}.$$

Proof: The proof of this lemma is standard and follows from local Torelli Theorem. ■

3.4 Construction of a Higgs Field on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathbf{M})/\mathcal{M}(\mathbf{M})}$

From now on we will consider only three dimensional CY manifolds.

Definition 30 *We define a Higgs field* $\tilde{\nabla}$ *on* $R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}$ *by using the Gauss-Manin connection and Poincare duality* Π *in the following manner:*

$$\tilde{\nabla}_{\phi_i}\Omega = \tilde{\nabla}_i\Omega := \Pi(\mathcal{D}_i(\Omega)) = \Pi(\Omega\lrcorner\phi_i), \tag{15}$$

where $\Omega \in H^1\left(M, \Omega^2\right)$ *and*$\{\phi_i\} \in H^1\left(M, T^{1,0}_M\right)$ *is an orthonormal basis with respect to the Weil-Petersson metric on* $T_0 \approx \mathbb{H}^1\left(M, T^{1,0}_M\right)$.

Lemma 31 *$\tilde{\nabla}$ as defined in Definition 30 is a Higgs field.*

Proof: We need to check that $\tilde{\nabla}$ satisfies the conditions in the definition 28 of a Higgs field. The definition (15) of $\tilde{\nabla}$ implies that

$$\tilde{\nabla} \in Hom\left(R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \otimes T(\mathcal{M}\,(\mathrm{M})\right), R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}). \tag{16}$$

On the other hand standard facts from commutative algebra imply

$$Hom\left(R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \otimes T(\mathcal{M}\,(\mathrm{M})\,, R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right) \backsimeq$$
$$\left(R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right)^* \otimes (T(\mathcal{M}\,(\mathrm{M})))^* \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \backsimeq$$
$$\backsimeq \left(R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right)^* \otimes \left(\Omega^1_{\mathcal{M}(\mathrm{M})} \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right) \backsimeq$$
$$\backsimeq Hom\left(R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\,, \Omega^1_{\mathcal{M}(\mathrm{M})} \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right). \tag{17}$$

So we obtain

$$Hom\left(R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \otimes T(\mathcal{M}\,(\mathrm{M})), R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right) \backsimeq$$
$$\backsimeq Hom\left(R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\,, \Omega^1_{\mathcal{M}(\mathrm{M})} \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right). \tag{18}$$

So the condition that

$$\tilde{\nabla} \in Hom\left(R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\,, R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \otimes \Omega^1_{\mathcal{M}(\mathrm{M})}\right)$$

follows directly from (16) and (18). Next we need to prove that $\tilde{\nabla} \circ \tilde{\nabla} = 0$. It is a standard fact that the relation $\tilde{\nabla} \circ \tilde{\nabla} = 0$ is equivalent to the relations $[\tilde{\nabla}_i, \tilde{\nabla}_j] = 0$. So in order to finish the proof of Lemma 31 we need to prove the following Proposition:

Proposition 32 *The commutator of $\tilde{\nabla}_i$ and $\tilde{\nabla}_j$ is equal to zero, i.e.*

$$[\tilde{\nabla}_i, \tilde{\nabla}_j] = 0. \tag{19}$$

Proof: Lemma 32 follows directly from the definition of $\tilde{\nabla}$. Indeed since the Gauss-Manin connection is a flat, i.e. $[\mathcal{D}_i, \mathcal{D}_j] = 0$ and the fact that the covariant derivative of Poincare duality is zero we deduce that:

$$\left[\tilde{\nabla}_i, \tilde{\nabla}_j\right] = [\Pi(\mathcal{D}_i), \Pi\,(\mathcal{D}_j)] = \Pi[\mathcal{D}_i, \mathcal{D}_j] = 0.$$

Proposition 32 is proved. ■ Lemma 31 is proved. ■

3.5 Higgs Field on the Tangent Bundle of $\mathcal{M}(\mathrm{M})$

Let $\omega_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ be the relative dualizing sheaf on $\mathcal{M}(\mathrm{M})$. We will define bellow a Higgs Field on the tangent vector bundle

$$T_{\mathcal{M}(\mathrm{M})} \approx \left(\omega_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right)^{*} \otimes R^{1}\pi_{*}\Omega^{2}_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$$

of the moduli space $\mathcal{M}(M)$. We denote by $\theta = id \otimes \tilde{\nabla}$, where $\tilde{\nabla}$ is defined by (15).

Lemma 33 *Let $\{\phi_i\}$ be a basis of the tangent bundle $\mathcal{T}_{\mathcal{M}(M)}$ restricted on some open polydisk $U \subset \mathcal{M}(M)$. We can identify the fibre $T^{1,0}_{M_\tau}$ of $\mathcal{T}_{\mathcal{M}(M)}$ with the harmonic forme $\mathbb{H}^1\left(M_\tau, T^{1,0}_{M_\tau}\right)$ with respect to the CY metric corresponding to the [p;arization class L. Let us define*

$$\vartheta : \mathcal{T}_{\mathcal{M}(M)} \to \mathcal{T}_{\mathcal{M}(M)}\left(\mathcal{M}(M)\right) \otimes \Omega^{1}_{\mathcal{M}(M)}\left(\mathcal{M}(M)\right)$$

by

$$\vartheta\left(\eta_\tau^{-1} \otimes \phi_j\right) := \eta_\tau^{-1} \otimes \sum_{i=1}^{N}\left(F^{-1}\left(\mathcal{D}_i(\phi_j \lrcorner \Omega_\tau)\right) \otimes (\phi_i)^{*}\right)$$

where $\mathcal{D}$ is the Gauss-Manin connection defined by (15), F is defined by (12) and η_τ is the holomorphic three form on M defined by Theorem 15. Then

$$\vartheta_i\left(\eta_\tau^{-1} \otimes \phi_j\right) = \eta_\tau^{-1} \otimes \left(\left((\phi_j \lrcorner \Omega_\tau)\lrcorner \phi_i\right)\lrcorner\left(\Omega_\tau\right)^{*}\right), \tag{20}$$

and ϑ defines a Higgs field on the tangent bundle of $\mathcal{M}(M)$.

Proof: Lemma 33 follows directly from Lemma 31, the definition of ϑ and the fact that

$$T_{\mathcal{M}(\mathrm{M})} \approx \left(\omega_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right)^{*} \otimes R^{1}\pi_{*}\Omega^{2}_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}.$$

Lemma 33 is proved. ■

We know that $h^{2,1} = \dim_{\mathbb{C}} H^1\left(\mathrm{M}, \Omega^2_{\mathrm{M}}\right) = h^{1,2} = \dim_{\mathbb{C}} H^2\left(\mathrm{M}, \Omega^1_{\mathrm{M}}\right)$ are constants. Therefore $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ and $R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ are holomorphic vector bundles over the moduli space $\mathcal{M}(\mathrm{M})$ of polarized CY threefold. We have a non degenerate pairing:

$$R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \times R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \to R^3\pi_*\Omega^3_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \tag{21}$$

given by

$$\sqrt{-1}\left((\phi_j(\tau)\lrcorner\Omega_\tau)\right) \wedge \omega_j(\tau) = h_{ij}(\tau)\Omega_\tau \wedge \overline{\Omega_\tau}, \tag{22}$$

where $(\phi_j(\tau)\lrcorner\Omega_\tau)$ and $\omega_j(\tau)$ are holomorphic sections of the holomorphic vector bundles $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ and $R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. Clearly $h_{ij}(\tau)$ depends holomorphically on $\tau \in \mathcal{M}(\mathrm{M})$. By using the non-degenerate pairing (21) we

can identify $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ with $R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ as follows: to the basis of holomorphic sections

$$\{\phi_i\lrcorner\Omega_{\tau_0}\}\in R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$$

we will assign

$$(\phi_i\lrcorner\Omega_{\tau_0})^*\in Hom\left(R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})},R^3\pi_*\Omega^3_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right)$$

such that for the pairing defined by (22) satisfies:

$$\phi_i\lrcorner\Omega_\tau\wedge(\phi_i\lrcorner\Omega_{\tau_0})^*=$$

$$\phi_i\lrcorner\Omega_\tau\wedge(\phi_i\lrcorner\Omega_{\tau_0})^*=\delta_{i,j}\Omega_\tau\wedge\overline{\Omega_\tau}. \tag{23}$$

So $(\phi_i\lrcorner\Omega_{\tau_0})$ and ω_l^* are given by the formula:

$$\omega_l^*=\sum_k h^{lk}\left(\phi_k\lrcorner\Omega_\tau\right) \tag{24}$$

$$(\phi_l\lrcorner\Omega_\tau)^*=\sum_k h^{kl}\omega_k. \tag{25}$$

Lemma 34 *Let us a fix a point $\tau_0\in\mathcal{M}(M)$. Suppose that*

$$\{\phi_i,\ i=1,...,N\}\ \text{ and }\ \{\omega_i,\ i=1,...,N\}$$

are some bases of $T_{\tau,\mathcal{M}(M)}(U)=\Omega_\tau\otimes R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}$ and $R^2\pi_*\Omega^1_{\mathcal{Y}(M)/\mathcal{M}(M)}$ in some open polydisk of τ_0. Let us define

$$C_{ijl}=-\sqrt{-1}\int_M(\Omega_\tau\wedge((\phi_i\wedge\phi_j\wedge\phi_l)\lrcorner\Omega_\tau)). \tag{26}$$

Let $h_{ij}:=\langle\phi_i\lrcorner\Omega_\tau,\omega_j\rangle$ be the pairing between the holomorphic vector bundles $R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}$ and $R^2\pi_*\Omega^1_{\mathcal{Y}(M)/\mathcal{M}(M)}$ defined by (22). Then

$$\vartheta_i\phi_j=\sum_{k,l=1}^N C_{ij}^k\left(\omega_k^*\lrcorner\Omega_\tau^*\right), \tag{27}$$

$$\vartheta_i\phi_j=\sum_{k=1}^N C_{ijk}\phi_k,$$

where C_{ij}^k and C_{ijk} are holomorphic functions in U. The relations between C_{ij}^k and C_{ijl} are given by

$$C_{ij}^k=\sum_{m=1}^n C_{ijm}h^{mk}, \tag{28}$$

and,

$$C_{ijl}=C_{jil}=C_{ilj}=C_{jli}=C_{lij}=C_{lij}. \tag{29}$$

Proof: (20) implies that

$$\vartheta_i\phi_j = ((\phi_j \lrcorner \Omega_\tau)\lrcorner\phi_i)\lrcorner(\Omega_\tau)^* = (\phi_i \wedge \phi_j \lrcorner \Omega_\tau)\lrcorner\Omega_\tau,$$

where $(\phi_i \wedge \phi_j \lrcorner \Omega_\tau) \in R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\mid_U$. Since $\{\omega_k\}$ is a basis of the holomorphic vector bundle $R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\mid_U$ we have

$$(\phi_i \wedge \phi_j \lrcorner \Omega_\tau) = \sum_k C^k_{ij}\omega_k. \tag{30}$$

Poincare duality and (30) imply that we can identify

$$(\phi_i \wedge \phi_j \lrcorner \Omega_\tau) \in R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$$

with the holomorphic section

$$\sum_k C^k_{ij}\omega^*_k \in R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})},$$

where $\omega^*_k \in R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\mid_U$ are defined by (24) and are the Poincare dual of $\omega_k \in R^2\pi_*\Omega^1_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\mid_U$. Thus we have

$$\langle \phi_l\lrcorner\Omega_\tau, \phi_i \wedge \phi_j \lrcorner \Omega_\tau\rangle = \left\langle \phi_l\lrcorner\Omega_\tau, \sum_k C^k_{ij}\omega^*_k \right\rangle =$$

$$\left\langle \phi_l\lrcorner\Omega_\tau, \sum_k C^k_{ij}\omega_k \right\rangle = \sum_k C^k_{ij}h_{lk}. \tag{31}$$

Combining (26) and (31) we get:

$$\left\langle \phi_l\lrcorner\Omega_\tau, \sum_k C^k_{ij}\omega_k \right\rangle = \langle \phi_l\lrcorner\Omega_\tau, \phi_i \wedge \phi_j \lrcorner \Omega_\tau\rangle =$$

$$-\sqrt{-1}\int\limits_{\mathrm{M}_{\tau_0}} (\Omega_\tau \wedge ((\phi_i \wedge \phi_j \wedge \phi_l)\lrcorner\Omega_\tau) = C_{ijl}. \tag{32}$$

So we can conclude from (22) and (25) that

$$C^k_{ij} = \sum_{l=1}^{N} C_{ijl}h^{lk}. \tag{33}$$

Thus (32) and (33) imply (27) and (28) .

Next we will prove (29) . We can multiply the global section η_τ of $\omega_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ defined in [31] by a constant and so we can assume that at the point $\tau_0 \in \mathcal{M}(\mathrm{M})$ we have

$$\eta_{\tau_0} = \Omega_0 \text{ and } \eta_\tau = \lambda(\tau)\Omega_\tau. \tag{34}$$

From the definition of ι given by (10) and formula (26), we conclude that (29) holds. Thus Lemma 34 is proved. ■

Lemma 35 *We have*

$$[\vartheta_i, \vartheta_j] = 0. \tag{35}$$

Proof: The formula for ϑ_i given by (20) implies

$$[\vartheta_i, \vartheta_j]\phi_k = F^{-1}([\mathcal{D}_i, \mathcal{D}_j](\phi_k \lrcorner \Omega_\tau)). \tag{36}$$

Since Gauss-Manin connection $\mathcal{D}$ is a flat connection then

$$[\mathcal{D}_i, \mathcal{D}_j] = 0. \tag{37}$$

Combining formula (37) with (36) we get $[\vartheta_i, \vartheta_j]\phi_k = 0$. Thus Lemma 35 is proved. ■

3.6 Relations with Frobenius Algebras

One can use Lemma 22 to define an associative product on the tangent bundle of the moduli space $\mathcal{M}(\mathrm{M})$ of three dimensional CY manifolds as follows: Let $\{\phi_i\}$ be a basis of $\mathrm{T}_{\tau_0, \mathcal{M}(\mathrm{M})} = H^1\left(\mathrm{M}, T^{1,0}_{\mathrm{M}}\right)$, then we define the product as:

$$\phi_i \times \phi_j = \mathfrak{i}^{-1}\left(\Pi\left(\mathcal{D}_i(\mathfrak{i}(\phi_j))\right)\right) = F_{ijk}\phi_k. \tag{38}$$

Lemma 36 *Let C^k_{ij} be defined by* (*26*)*, then*

$$F_{ijk} = \sqrt{-1} C_{ijk}. \tag{39}$$

Proof: Lemma 36 follows directly from the formulas for F_{ijk} and C_{ijk}. ■

Corollary 37 *The relations* (*29*) *and* (*35*) *shows that F_{ijk} define a structure of a commutative algebra on the tangent bundle of the moduli space $\mathcal{M}(M)$ of three dimensional CY manifolds.*

3.7 The Analogue of Cecotti-Vafa tt* Equations on $\mathcal{M}(\mathrm{M})$

Definition 38 *Let ϑ be the Higgs field defined by* (*20*)*. Theorem 14 implies the existence of a global holomorphic non vanishing section η_τ of the line bundle $\pi_*\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}$. Then η_τ defines a metric on the flat line bundle $\pi_*\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}$ with curvature zero. Let us define the Weil-Petersson metric on*

$$\mathcal{T}_{\mathcal{M}(M)} \approx Hom\left(\pi_*\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}, R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}\right) \approx$$

$$\left(\pi_*\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}\right)^* \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}$$

as follows: Let

$$\phi_\tau = (\eta_\tau)^* \otimes \omega_\tau(1,1) \in \mathcal{T}_{\mathcal{M}(M)} \approx \left(\pi_*\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}\right)^* \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}.$$

Let us define the function

$$\lambda(\tau) := \frac{\eta_\tau}{\Omega_\tau}. \tag{40}$$

Then

$$\|\phi_\tau\|^2 := |\lambda(\tau)|^{-2} \otimes \sqrt{-1}\int\limits_M \omega_\tau(1,2) \wedge \overline{\omega_\tau(1,2)} =$$

$$\frac{\sqrt{-1}\left\langle \omega_\tau(1,2), \overline{\omega_\tau(1,2)} \right\rangle}{|\lambda(\tau)|^2}. \tag{41}$$

Let ∇ be the standard connection of the metric defined by (41). We will define the Cecotti-Hitchin-Vafa-Simpson (CHVS) connection $D = D + \overline{D}$ on the tangent bundle of the moduli space $\mathcal{M}(M)$ of three dimensional CY manifolds as follows:

$$D_i : \nabla_i + t\vartheta_i \text{ and } D_{\overline{j}} = \nabla_{\overline{j}} + t^{-1}\overline{\vartheta}_j, \tag{42}$$

where $t \in \mathbb{C}^*$.

Theorem 39 *The curvature of the metric defined by (41) in the flat coordinates defined by Definition 10 is given by:*

$$R_{i\overline{j},k\overline{l}} = -\sqrt{-1}\int\limits_M ((\phi_i \wedge \phi_k) \lrcorner \Omega_M) \wedge \left(\overline{(\phi_j \wedge \phi_l) \lrcorner \Omega_M}\right) =$$

$$= -\sqrt{-1}\left\langle (\phi_i \wedge \phi_k) \lrcorner \Omega_M, (\phi_j \wedge \phi_l) \lrcorner \Omega_M \right\rangle. \tag{43}$$

Proof: Since the metric on $(\pi_*\omega_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})})^*$ is flat, then it has a zero curvature. The connection of the metric defined by (41) will be

$$\nabla := \widetilde{\nabla} \otimes id \oplus id \otimes \nabla_1,$$

where $\widetilde{\nabla}$ is the flat connection on $\left(\pi_*\omega_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right)^*$ and ∇_1 is the connection on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. Thus we get that for the curvature $[\nabla, \nabla]$ we have that $[\nabla, \nabla] = [\nabla_1, \nabla_1]$. So the curvature of the metric defined by (41) is equal equal to the curvature on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. Formula (7) implies

$$\sqrt{-1}\partial_i\overline{\partial}_{\overline{j}}\left(\langle \Omega_\tau, \Omega_\tau \rangle\right) =$$

$$\sqrt{-1}\left\langle \Omega_0 \lrcorner \phi_i, \Omega_0 \lrcorner \phi_j \right\rangle - \frac{\sqrt{-1}}{2}\sum_{k,l}\left\langle ((\phi_i \wedge \phi_k) \lrcorner \Omega_0)), (\phi_j \wedge \phi_l) \lrcorner \Omega_0 \right\rangle \tau^k \overline{\tau}^l + O(|\tau|^3). \tag{44}$$

Thus (44) implies (43). Theorem 39 is proved. ■

We will show that the Cecotti-Hitchin-Vafa-Simpson connection is flat.

Theorem 40 *The connection D defined by (42) is a flat one, i.e.:*

$$[D_i, D_j] = [D_i, D_{\overline{j}}] = [D_{\overline{i}}, D_{\overline{j}}] = 0 \tag{45}$$

for all $0 \leqq i, j \leqq N$.

Proof: First we will prove that $[D_i, D_j] = 0$. The definition of $D_i = \nabla_i + t\vartheta_i$ implies that

$$[D_i, D_j] = [\nabla_i, \nabla_j] + t[\nabla_i, \vartheta_j] + t^2[\vartheta_i, \vartheta_j]. \tag{46}$$

We know that $\triangledown_i$ is a Hermitian connection of the Weil-Petersson metric defined by (41), which is a Kähler metric and thus the (2,0) part of its curvature is zero. This implies that $[\nabla_i, \nabla_j] = 0$. From Lemma 34 we know that $[\vartheta_i, \vartheta_j] = 0$.

Lemma 41 *We have*

$$[\nabla_i, \vartheta_j] = 0. \tag{47}$$

Proof: In the flat coordinates $(\tau^1, ..., \tau^N)$ at a fixed point $\tau_0 = 0 \in \mathcal{M}(\mathrm{M})$ of the moduli space we have that $\nabla_i = \partial_i$. The definition of ϑ_i given by the formula (20) and applied to

$$\phi_k(\tau) := \left(\frac{\partial}{\partial \tau^k}\Omega_\tau\right) \lrcorner \left(\Omega_\tau\right)^*, \; k = 1, .., N, \tag{48}$$

where Ω_τ is defined by (6) gives that we have at the point $\tau_0 = 0$:

$$\nabla_i \left(\vartheta_j \left(\phi_k(\tau)\right)\right)|_{\tau=\tau_0} =$$

$$\vartheta_i \left(\nabla_j \left(\phi_k(\tau)\right)\right) = \left(\left(\frac{\partial^2}{\partial \tau^i \partial \tau^j}\left(\frac{\partial}{\partial \tau^k}\Omega_\tau\right)\right) \lrcorner \left(\Omega_\tau\right)^*\right). \tag{49}$$

So (49) implies (47). Lemma 41 is proved. ∎

Corollary 42 $[D_i, D_j] = [D_{\overline{i}}, D_{\overline{j}}] = 0.$

Lemma 43 *We have* $[\nabla_i + t\vartheta_i, \nabla_{\overline{j}} + t^{-1}\overline{\vartheta_j}] = 0.$

Proof: We will identify $T_{0,\mathcal{M}(\mathrm{M})}$ with $H^1\left(\mathrm{M}_0, \Omega^2_{\mathrm{M}_0}\right)$ as in Proposition 23. We assumed that $\{\phi_i\}$ given by (48) is an orthonormal basis of the tangent space $T_{0,\mathcal{K}} = H^1\left(\mathrm{M}, T^{1,0}_{\mathrm{M}}\right)$ at at point $0 \in \mathcal{M}(\mathrm{M})$. We have in the flat coordinates $(\tau^1, ..., \tau^N)$ that $\phi_k(\tau) = \left(\nabla_k\Omega_\tau\right) \lrcorner \eta_\tau^{-1}\right)$. We will need the following Propositions

Proposition 44 *We have*

$$\langle \nabla_i\phi_k, \nabla_j\phi_l\rangle = \frac{1}{|\lambda(\tau)|^2}\int\limits_M \left(\left(\phi_i \wedge \phi_k\right)\lrcorner\Omega_\tau\right) \wedge \overline{\left(\left(\phi_j \wedge \phi_l\right)\lrcorner\Omega_\tau\right)}, \tag{50}$$

where $(\phi_i \wedge \phi_k)\lrcorner\Omega_\tau$ *and* $(\phi_j \wedge \phi_l)\lrcorner\Omega_\tau$ *are forms of type* $(1,2)$ *on* M_τ.

Proof: Since $\phi_i(\tau) = \left(\left(\nabla_i\Omega_\tau\right)\lrcorner\eta_\tau^{-1}\right)$ we get that

$$\langle \nabla_i\phi_k, \nabla_j\phi_l\rangle|_{\tau=0} = \left\langle\left(\nabla_i\nabla_k\Omega_\tau\right)\lrcorner\eta_\tau^{-1}, \left(\nabla_j\nabla_l\Omega_\tau\right)\lrcorner\eta_\tau^{-1}\right\rangle|_{\tau=0}.$$

From (13) we derive that

$$\nabla_i\nabla_k\Omega_\tau|_{\tau=0} = (\phi_i \wedge \phi_k)\lrcorner\Omega_0 \text{ and } \nabla_j\nabla_l\Omega_\tau|_{\tau=0} = (\phi_j \wedge \phi_l)\lrcorner\Omega_0$$

are form of type $(1,2)$. Thus we get

$$\left\langle \nabla_i \phi_k, \nabla_j \phi_l \right\rangle|_{\tau=0} = \left(\frac{1}{|\lambda(\tau)|^2} \int_{\mathrm{M}} ((\phi_i \wedge \phi_k) \lrcorner \Omega_\tau) \wedge \overline{((\phi_j \wedge \phi_l) \lrcorner \Omega_\tau)} \right)|_{\tau=0} ,$$

where $(\phi_i \wedge \phi_k) \lrcorner \Omega_0$ and $(\phi_j \wedge \phi_l) \lrcorner \Omega_0$ are forms of type $(1,2)$. Proposition 44 is proved. $\blacksquare$

Proposition 45 *We have*

$$\vartheta_i(\phi_k(\tau))|_{\tau=0} = \left((\Pi\left((\phi_i(\tau) \wedge \phi_k(\tau)) \lrcorner \Omega_\tau \right)) \lrcorner (\Omega_\tau)^{-1} \right)|_{\tau=0}$$

and

$$\left\langle \vartheta_i \phi_k, \vartheta_j \phi_l \right\rangle|_{\tau=0} =$$

$$\left(\frac{1}{|\lambda(\tau)|^2} \int_M (\Pi\left((\phi_i(\tau) \wedge \phi_k(\tau)) \lrcorner \Omega_\tau \right)) \wedge \overline{(\Pi\left((\phi_j(\tau) \wedge \phi_l(\tau)) \lrcorner \Omega_\tau \right))} \right)|_{\tau=0} , \quad (51)$$

where $\Pi\left((\phi_i \wedge \phi_k) \lrcorner \Omega_0 \right)$ *and* $\Pi\left((\phi_j \wedge \phi_l) \lrcorner \Omega_0 \right)$ *are forms of type* $(2,1)$.

Proof: It follows from the definition of ϑ

$$\vartheta_i(\left(\nabla_k \Omega_\tau\right) \lrcorner \Omega_\tau^{-1})|_{\tau=0} = \left((\Pi\left(\phi_i \wedge \phi_k \lrcorner \Omega_0\right)) \lrcorner \Omega_0^{-1} \right), \quad (52)$$

where $\Pi\left(\phi_i \wedge \phi_k \lrcorner \Omega_\tau\right)$ is the Poincare dual of the form $(\phi_i \wedge \phi_k \lrcorner \Omega_0)$ of type $(1,2)$. Thus $\Pi\left(\phi_i \wedge \phi_k \lrcorner \Omega_0\right)$ is a form of type $(2,1)$ and (52) implies (51). Proposition 45 is proved. $\blacksquare$

Proposition 46 *We have the following formula:*

$$\left\langle ((\phi_i \wedge \phi_k) \lrcorner \Omega_0), \overline{((\phi_j \wedge \phi_l) \lrcorner \Omega_0)} \right\rangle = \int_M ((\phi_i \wedge \phi_k) \lrcorner \Omega_0) \wedge \overline{((\phi_j \wedge \phi_l) \lrcorner \Omega_0)} =$$

$$- \left\langle \Pi((\phi_i \wedge \phi_k) \lrcorner \Omega_0), \overline{\Pi((\phi_j \wedge \phi_l) \lrcorner \Omega_0)} \right\rangle =$$

$$= - \int_M \Pi((\phi_i \wedge \phi_k) \lrcorner \Omega_0) \wedge \overline{\Pi((\phi_j \wedge \phi_l) \lrcorner \Omega_0)}. \quad (53)$$

Proof: Since $\{\phi_i\}$ is an orthonormal basis in $T_{0,\mathcal{M}(\mathrm{M}_0)}$ with respect to the W.-P. metric we get that $\{\phi_i \lrcorner \Omega_0\}$ is orthonormal basis in $H^1(\mathrm{M})$ and $\{\overline{\phi_i \lrcorner \Omega_0}\}$ is an orthonormal basis in $H^2\left(\mathrm{M}_0, \Omega^1_{\mathrm{M}_0}\right)$. Thus

$$\sqrt{-1} \left\langle \phi_i \lrcorner \Omega_0, (\phi_j \lrcorner \Omega_0) \right\rangle = \sqrt{-1} \int_{\mathrm{M}} (\phi_i \lrcorner \Omega_0) \wedge \overline{(\phi_j \lrcorner \Omega_0)} = \delta_{i\overline{j}} \quad (54)$$

and

$$\sqrt{-1}\left\langle \overline{\phi_i \lrcorner \Omega_0}, \overline{\phi_j \lrcorner \Omega_0}\right\rangle = \sqrt{-1}\int\limits_{\mathrm{M}} \overline{\phi_i \lrcorner \Omega_0} \wedge \phi_j \lrcorner \Omega_0 = -\delta_{i\overline{j}}. \tag{55}$$

Let

$$(\phi_i \wedge \phi_k) \lrcorner \Omega_0 = \sum_{\nu=1}^{N} \alpha_\nu \overline{\phi_\nu \lrcorner \Omega_0},\ (\phi_j \wedge \phi_l) \lrcorner \Omega_0 = \sum_{i=1}^{N} \beta_\mu \left(\overline{\phi_\mu \lrcorner \Omega_0}\right), \tag{56}$$

then

$$\Pi\left((\phi_i \wedge \phi_k) \lrcorner \Omega_0\right) = \sum_{\nu=1}^{N} \alpha_\nu \left(\phi_\nu \lrcorner \Omega_0\right) \text{ and } \Pi\left((\phi_j \wedge \phi_l) \lrcorner \Omega_0\right) = \sum_{i=1}^{N} \beta_\mu \left(\phi_\mu \lrcorner \Omega_0\right). \tag{57}$$

Combining (54), (55), (56) and (57) we get that

$$\sqrt{-1}\left\langle ((\phi_i \wedge \phi_k) \lrcorner \Omega_0), ((\phi_j \wedge \phi_l) \lrcorner \Omega_0)\right\rangle = \sum_{\nu=1}^{N} \alpha_\nu \beta_\nu \tag{58}$$

and

$$\sqrt{-1}\left\langle \Pi((\phi_i \wedge \phi_k) \lrcorner \Omega_0), (\Pi(\phi_j \wedge \phi_l) \lrcorner \Omega_0)\right\rangle = -\sum_{\nu=1}^{N} \alpha_\nu \beta_\nu. \tag{59}$$

Thus (58) and (59) imply Proposition 46. ■

We have

$$\left\langle \left[\nabla_i + t\vartheta_i, \nabla_{\overline{j}} + t^{-1}\overline{\vartheta_j}\right] \phi_k, \phi_l \right\rangle = \left\langle \left[\nabla_i, \nabla_{\overline{j}}\right] \phi_k, \phi_l \right\rangle + \left\langle \left[\vartheta_i, \overline{\vartheta_j}\right] \phi_k, \phi_l \right\rangle. \tag{60}$$

From (43) we derive that

$$R_{i\overline{j}.k\overline{l}} = \left\langle [\nabla_i, \nabla_{\overline{j}}]\phi_k, \phi_l \right\rangle = -\frac{\sqrt{-1}}{|\lambda(0)|^2}\left\langle ((\phi_i \wedge \phi_k) \lrcorner \Omega_0)), (\phi_j \wedge \phi_l) \lrcorner \Omega_0 \right\rangle. \tag{61}$$

(51) implies

$$\left\langle \left[\vartheta_i, \overline{\vartheta_j}\right] \phi_k, \phi_l \right\rangle = \frac{1}{|\lambda(0)|^2}\left\langle \left(\Pi\left((\phi_i \wedge \phi_k) \lrcorner \Omega_0\right)\right), \overline{\left(\Pi\left((\phi_j \wedge \phi_l) \lrcorner \Omega_0\right)\right)} \right\rangle. \tag{62}$$

Combining (60), (61), (62) with (53) we conclude that $[\nabla_i + t\vartheta_i, \nabla_{\overline{j}} + t^{-1}\overline{\vartheta_j}] = 0$. Theorem 40 is proved. ■

Corollary 47 *The connection constructed in Theorem 40 is a flat $\mathbb{S}p(2h^{2,1}, \mathbb{R})$ connection on the tangent space of the moduli space $\mathcal{M}(M)$ of three dimensional CY manifolds. The imaginary form of the Weil-Petersson metric is a parallel form with respect to the CHSV connection.*

Proof: It is an well known fact that the imaginary part of the Weil-Petersson metric $\omega_\tau(1,1)$ is parallel with respect to the CHSV connection since it is the imaginary part of a Kähler metric. On the other hand $\omega_\tau(1,1)$ is just the restriction of the intersection form on $H^3(\mathrm{M})$ and so it is parallel with respect to the Gauss-Manin connection and so with respect to the connection θ. From here Corollary 47 follows directly. ■

Remark 48 *It is easy to see that Cecotti-Vafa tt* equations are exactly the Hitchin-Simpson self duality equations studied in [17], [26] and [27].*

We will call the connection that we constructed a Cecotti-Hitchin-Simpson-Vafa connection and will denote it as a CHSV connection.

4 Review of the Geometric Quantization

4.1 Basic Notions of ADW Geometric Quantization

In this paragraph we are going to review the method of the geometric quantization described in [2] and [37]. We will consider a linear space $\mathbf{W} \approxeq \mathbb{R}^{2n}$ with a constant symplectic structure

$$\omega = \frac{1}{2}\omega_{ij} dt^i \wedge dt^j, \tag{63}$$

where ω_{ij} is a constant invertible matrix and the x^i linear coordinates on $\mathbb{R}^{2n} = \mathbf{W}$. We will denote by ω^{-1} the matrix inverse to ω and obeying $\omega_{ij}\left(\omega^{-1}\right)^{jk} = \delta_i^k$.

Definition 49 *The "prequantum line bundle" is a unitary line bundle $\mathcal{L}$ over $\mathbf{W}$ with a connection whose curvature is $\sqrt{-1}\omega$. Up to an isomorphism, there is only one such choice of $\mathcal{L}$. One can take $\mathcal{L}$ to be the trivial unitary line bundle, with a connection given by the covariant derivatives*

$$\frac{D}{Dt^i} = \frac{\partial}{\partial t^i} + \frac{\sqrt{-1}}{2}\omega_{ij} t^j. \tag{64}$$

Definition 50 *We define the $\mathbf{L}^2$ norm of the sections of $\mathcal{L}$ as follows; Let h be a positive function on $\mathbf{W}$ which define a metric on $\mathcal{L}$ and $dd^c \log h = \omega$. Then we will say that the $\mathbf{L}^2$ norm of a section ϕ of $\mathcal{L}$ is defined as*

$$\|\phi\|^2_{\mathbf{L}^2} = (-1)^{n(n-1)/2}\left(-\frac{\sqrt{-1}}{2}\right)^n \int\limits_{\mathbf{W}} \exp\left(-h\right)\left|\phi\right|^2 dz^1 \wedge ... \wedge dz^n \wedge \overline{dz^1 \wedge ... \wedge dz^n}. \tag{65}$$

Then we define the"prequantum Hilbert space" $\mathcal{H}_0$ as the Hilbert space that consists of sections of $\mathcal{L}$ with a finite $\mathbf{L}^2$ norm.

In order to define the quantum Hilbert space, we will introduce the notion of polarization.

Definition 51 *We will define the polarization as a choice of a complex structure J on $\mathbf{W}$ with the following properties: **a.** J is a translation invariant, so it is defined by a constant matrix J^i_j with $J^2 = -1$. **b.** The two-form ω is of type (1,1) with respect to the complex structure J. **c.** J is positive in the sense that the bilinear form g defined by $g(u,v) = \omega(u, Jv)$ is strictly positive.*

Definition 52 *Given such a complex structure J, we define the quantum Hilbert space H_J to be space of all holomorphic functions $\phi_J(z^1, ..., z^n)$ on $\mathbf{W}$ with respect to the complex structure J with a finite $\mathbf{L}^2$ norm.*

It is well known that the Heisenberg group of $\mathbf{W}$ has an irreducible projective representation in $\mathcal{H}_J$. Thus each such representation of the Heisenberg group of $\mathbf{W}$ depends on the choice of the complex structure J of $\mathbf{W}$. We want to construct a projectively flat connection on the infinite dimensional dimensional vector space over the parameter space of the complex structures of $\mathbf{W}$. Construction of such a connection enables one to identify all the irreducible projective representation of the Heisenberg group in $\mathcal{H}_J$.

4.2 Siegel Space

We will introduce some notations following [37]. First of all, one has the projection operators $\frac{1}{2}(1 \mp \sqrt{-1}J)$ on $W^{1,0}$ and on $\overline{\mathbf{W}^{1,0}} = W^{0,1}$, where

$$\mathbf{W}^{1,0} := \{u \in \mathbf{W} \otimes \mathbb{C} \,\big|\, Ju = \sqrt{-1}u\} \ .$$

We know that the vector space W with a complex structure J can be identified as a complex vector space canonically with the spaces $W^{1,0}$ and $W^{0,1}$ by the maps

$$u \to \frac{1}{2}(1 \mp \sqrt{-1}J)u.$$

We need to write down explicitly in fixed coordinates these two identifications. We will follow the notations from [37] in the above identifications. For any vector $v = (..., v^i, ...)$, we denote by

$$v^{\underline{i}} = \frac{1}{2}(1 - \sqrt{-1}J)^i_j v^j \text{ and } v^{\overline{i}} = \frac{1}{2}(1 + \sqrt{-1}J)^i_j v^j.$$

For one forms $w = (..., w_i, ...)$ we have

$$w_{\underline{j}} = \frac{1}{2}(1 - \sqrt{-1}J)^i_j w_i \text{ and } w_{\overline{j}} = \frac{1}{2}(1 + \sqrt{-1}J)^i_j w_j.$$

Thus $J^{\underline{i}}_{\underline{j}} = \sqrt{-1}\delta^{\underline{i}}_{\underline{j}}$ and $J^{\overline{i}}_{\overline{j}} = \sqrt{-1}\delta^{\overline{i}}_{\overline{j}}$. This means that the projections of J^i_j and δ^i_j on $W^{1,0}$ and $W^{0,1}$ are proportional.

Let $\mathfrak{Z}$ be the space of all J obeying the conditions in Definition 51. Then $\mathfrak{Z}$ is a symmetric space, i.e. $\mathfrak{Z} = \mathbb{S}p(2n, \mathbb{R})/\mathbb{U}(n)$. $\mathfrak{Z}$ is called Siegel space. It is a well known fact that we have the following realization of $\mathfrak{Z} = \mathbb{S}p(2n, \mathbb{R})/\mathbb{U}(n)$ as a tube domain;

$$\mathfrak{Z} = \{Z | Z \textit{ is a } (n \times n) \text{ complex matrix such that } Z^t = Z \textit{ and } \operatorname{Im} Z > 0\}.$$

$\mathfrak{Z}$ has a natural complex structure, defined as follows. The condition $J^2 = -1$ implies that for a first order variation δJ of J, one must have

$$J \circ \delta J + \delta J \circ J = 0$$

This means that the non-zero projections of δJ on (1,0) vectors and (0,1) vectors are $\delta J^{\underline{i}}_{\bar{j}}$ and $\delta J^{\bar{i}}_{\underline{j}}$. We define the complex structure on $\mathfrak{Z}$ by declaring $\delta J^{\underline{i}}_{\bar{j}}$ to be of type $(1,0)$ and $\delta J^{\bar{i}}_{\underline{j}}$ to be of type $(0,1)$. Notice that the projection of δJ on $(0,1)$ vectors is a map from $(1,0)$ vectors to $(0,1)$ vectors. So the $(1,0)$ part of δJ is the Beltrami differential as it was defined in **Section 2** of this article.

4.3 Construction of Witten Projective Connection

Over $\mathfrak{Z}$ we introduce two Hilbert space bundles. One of them, say $\mathcal{H}^0$, is the trivial bundle $\mathfrak{Z}\times\mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of all function $\psi(t^i;J)$ with finite $\mathbf{L}^2$ norm. The definition of $\mathcal{H}_0$ is independent of J. The second is the bundle $\mathcal{H}^Q$, whose fibre over a point $J\in\mathfrak{Z}$ are functions $\psi(t^i;J)$ of t^i, holomorphic in the complex structure defined by J, i.e. the following equation holds:

$$\frac{D}{Dt^{\bar{i}}}\psi(t^i;J)=0.$$

This equation has a dependence on J coming from the projection operators used in defining $t^{\bar{i}}$.

A connection on the bundle $\mathcal{H}^0$ restricts to a connection on $\mathcal{H}^Q$ if and only if its commutator with $D_{\bar{i}}$ is a linear combination of the $D_{\bar{j}}$. Since $\mathcal{H}^0$ is defined as a product bundle $\mathfrak{Z}\times\mathcal{H}_0$, there is a trivial connection δ on this bundle

$$\delta := \sum_{i,j} dJ^i_j\frac{\partial}{\partial J^i_j}. \tag{66}$$

We can expand δ in $(1,0)$ and $(0,1)$ pieces, $\delta=\delta^{(1,0)}+\delta^{(0,1)}$, with

$$\delta^{(1,0)}=\sum_{i,j} dJ^{\underline{i}}_{\bar{j}}\frac{\partial}{\partial J^{\underline{i}}_{\bar{j}}}. \tag{67}$$

Unfortunately, as it was shown in [37] the commutator of $\delta^{(1,0)}$ with $D_{\bar{i}}$ is not a linear combination of the $D_{\bar{j}}$. So one needs to modify the trivial connection so that its commutator with $D_{\bar{i}}$ will be a linear combination of the $D_{\bar{j}}$

Definition 53 *Witten defined in [37] the following connection ∇ on the bundle $\mathcal{H}^0=\mathfrak{Z}\times\mathcal{H}_0\to\mathfrak{Z}$:*

$$\nabla^{(1,0)} := \delta^{(1,0)}-\frac{1}{4}(dJ\omega^{-1})^{\underline{ij}}\frac{D}{Dt^{\underline{i}}}\frac{D}{Dt^{\underline{j}}} \text{ and } \nabla^{(0,1)} := \delta^{(0,1)}. \tag{68}$$

Witten proved the following theorem in [37]:

Theorem 54 ***A.*** *The connection ∇ descends to a connection on $\mathcal{H}^Q$.* ***B.*** *The curvature of the connection ∇ on $\mathcal{H}^Q$ is of type $(1,1)$ and it is equal to*

$$[\nabla^{(0,1)},\nabla^{(1,0)}]=\left[\delta^{(0,1)},-\frac{1}{4}(dJ\omega^{-1})^{\underline{ij}}\frac{D}{Dt^{\underline{i}}}\frac{D}{Dt^{\underline{j}}}\right]=-\frac{1}{8}dJ^{\underline{i}}_{\bar{k}}dJ^{\bar{k}}_{\underline{j}}. \tag{69}$$

This theorem shows that the curvature of ∇ is not zero, even when it is restricted to $\mathcal{H}^Q$. The curvature is a c-number, that is, it depends only on J, and not on the variables t^i that are being quantized. The fact that the curvature of ∇ is a c-number means that parallel transport by ∇ is unique up to a scalar factor which, moreover, it is of modulus 1 since the curvature is real or more fundamentally since ∇ is unitary. So up to this factor one can identify the various $\mathcal{H}_J$'s, and regard them as a different realization of the quantum Hilbert space $\mathcal{H}$.

5 Geometric Quantization of Moduli Space of CY Manifolds and Quantum Background Independence

5.1 Symplectic Structures and the Flat Coordinates

Our goal is to quantize geometrically the cotangent bundle of the moduli space $\mathcal{M}\,(\mathrm{M})$. This means to define a flat $\mathbb{S}p(2h^{2,1},\mathbb{R})$ connection on the cotangent bundle of $\mathcal{M}\,(\mathrm{M})$. We have done this in the **Section 3.** Then will construct the prequantum line bundle. Once the prequatum line bundle is constructed we will compute explicitly the projective connection defined in [2] and [37]. After the geometric quantization is done we will derive BCOV holomorphic anomaly equations established in [4] as the projective flat connections on the Hilbert bundle $\mathcal{H}^Q$. To perform these computations, it is important to fix first the flat symplectic structure on the tangent bundle $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$, then the local coordinates on $\mathcal{M}\,(\mathrm{M})$ that describe the change of the complex structures on the CY manifold M_τ and the linear coordinates in $W^*_\tau = \Omega^1_{\tau,\mathcal{M}(\mathrm{M})}$.

Remark 55 *We will define the symplectic form $\omega_1(\tau)$ on the vector bundle*

$$R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)} \to \mathcal{M}(M)$$

as the imaginary part of the metric

$$g_{i,\overline{j}} := \left\langle \frac{\partial}{\partial\tau^i}\Omega_\tau, \frac{\partial}{\partial\tau^j}\Omega_\tau \right\rangle = \sqrt{-1}\int\limits_M \frac{\partial}{\partial\tau^i}\Omega_\tau \wedge \overline{\frac{\partial}{\partial\tau^j}\Omega_\tau}\ . \tag{70}$$

The symplectic structure on the tangent space $\mathcal{T}_{\mathcal{M}(M)}$ is defined by the imaginary part of the Weil-Petersson metric, i.e.:

$$\omega(\tau) = \operatorname{Im} G\left|_{T_{\tau,\mathcal{K}}}\right. , \tag{71}$$

where

$$G_{i,\overline{j}} := \left\langle \left(\frac{\partial}{\partial\tau^i}\Omega_\tau\right)\lrcorner\eta_\tau^{-1}, \left(\frac{\partial}{\partial\tau^j}\Omega_\tau\right)\lrcorner\eta_\tau^{-1} \right\rangle_{WP} = \frac{\sqrt{-1}}{\|\eta_\tau\|^2}\int\limits_M \frac{\partial}{\partial\tau^i}\Omega_\tau \wedge \overline{\frac{\partial}{\partial\tau^j}\Omega_\tau}\ . \tag{72}$$

Thus (72) implies the following relations between the two symplectic structures:

$$g_{i,\overline{j}} = e^{-K} G_{i,\overline{j}}, \tag{73}$$

where

$$\|\eta_\tau\|^{-2} = -\sqrt{-1}\int_M \eta_\tau \wedge \overline{\eta_\tau} = \exp(-K). \tag{74}$$

We know from Remark 48 that the forms η_τ are parallel with respect to the Cecotti-Hitchin-Simpson-Vafa connection. Thus we can identified the symplectic structures of the tangent bundle by the flat $\mathbb{S}p(2h^2, \mathbb{R})$ Cecotti-Hitchin-Simpson-Vafa connection.

Remark 56 *We know that if we fix a basis of the orthogonal vectors $\{\phi_{i,\tau}\}$ in the holomorphic tangent space*

$$T_{\tau,\mathcal{K}} = H^1\left(M_\tau, T^{1,0}_{M_{gt}}\right)$$

then we obtain a linear coordinate system $(\tau^1, ..., \tau^N)$ in the dual $\Omega^1_{\tau,\mathcal{K}}$ of $T_{\tau,\mathcal{K}}$. $\Omega^1_{\tau,\mathcal{K}}$ can be canonically identified with $T_{\tau,\mathcal{K}}$ by using the parallel symplectic form. According to the results obtained in [32] then linear coordinate system $(\tau^1, ..., \tau^N)$ defined by the choice of the orthogonal vectors $\{\phi_{i,\tau_0}\}$ in the holomorphic tangent space

$$T_{\tau_\tau,\mathcal{K}} = H^1\left(M_{\tau_0}, T^{1,0}_{M_{\tau_0}}\right)$$

defines the same local coordinate system $(\tau^1, ..., \tau^N)$ in $\mathcal{K}$ since

$$\mathcal{K} \subset H^1\left(M, T^{1,0}_{M_{\tau_0}}\right).$$

See Theorem 6.

Remark 57 *In [32] we construct in a canonical way a family of holomorphic form Ω_τ given by (6). Then the family of holomorphic form Ω_τ defines a choice of a basis*

$$\{\phi_{i,\tau} \lrcorner \Omega_\tau\} \text{ and } \{\overline{(\phi_{i,\tau} \lrcorner \Omega_\tau)}\} \tag{75}$$

of the complexified tangent space: $H^1\left(M, \Omega^2_M\right) \oplus H^2\left(M, \Omega^1_M\right)$. Thus the flat local coordinate system $(\tau^1, ..., \tau^N)$ is defined by a choice of the basis in the holomorphic tangent space $T_{\tau,\mathcal{K}}$ defines in a canonical way a coordinate system on $H^1(M)$ which is the same as $(\tau^1, ..., \tau^N)$ in case we choose the basis (75). We identified the dual of $H^1(M,)$ with $H^1(M)$ by using the parallel symplectic form.

Remark 58 *The complex structure J_τ on $T_{\tau,\mathcal{M}(M)}$ is defined by the complex structure on $H^1\left(M, T^{1,0}_M\right)$. Indeed by local deformation theory we have*

$$T^{1,0}_{\tau,\mathcal{M}(M)} = H^1\left(M, T^{1,0}_M\right).$$

On CY manifold $H^1(M_\tau, T_\tau^{1,0})$ *can be identified with* $H^1\left(M,\Omega_M^2\right)$ *by contraction with the non zero holomorphic form* Ω_τ. *Thus the complex structure operator* J_τ *on the CY manifolds acts on*

$$T^{1,0}_{\tau,\mathcal{M}(M)} = H^1\left(M,T_M^{1,0}\right)$$

in a natural way as follows

$$J\left((dz^i) \wedge (J\left(dz^j\right) \wedge (J\left(\overline{dz}^k\right)\right) = \sqrt{-1}(dz^i \wedge dz^j \wedge \overline{dz}^k)$$

since $J(dz^i) = \sqrt{-1}dz^i$.

Remark 59 *We are using two different identifications of* $H^1(M_\tau, T^{1,0}_{M_\tau})$ *with* $H^1\left(M,\Omega_M^2\right)$ *by using using the contraction with the families of the two holomorphic* $3-$*forms* η_τ *and* Ω_τ. *Recall that* η_τ *was defined globally by Theorem 15. We obtained two coordinate systems* $(\tau^1, ..., \tau^N)$ *and* $(t^1, ..., t^N)$ *on* $H^1\left(M,\Omega_M^2\right)$. *The relation between them is given by the relations between* Ω_τ *and* η_τ, *i.e. the two coordinate systems are proportional with the coefficients of proportionality* $\lambda(\tau)$, *where* $\Omega_\tau = \lambda(\tau)\eta_\tau$.

Next we will define the prequantum line bundles over $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ and $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$.

Theorem 60 *The prequantum line bundles on the tangent bundle* $\mathcal{T}_{\mathcal{M}(M)}$ *is* $p^*\left(\pi_*\left(\omega^*_{\mathcal{Y}(M)/\mathcal{M}(M)}\right)\right)$, *where* $p : \mathcal{T}_{\mathcal{M}(M)} \rightarrow \mathcal{M}\left(M\right)$, $\omega_{\mathcal{Y}(M)/\mathcal{M}(M)}$ *is the relative dualizing line bundle and the metric on its fibre is defined by the* L^2 *norm of the holomorphic three form. The Chern class of* $\pi^*\left(\omega^*_{\mathcal{Y}(M)\ /\mathcal{M}(M)}\right)$ *is given by the restriction of*

$$\omega(\tau) := \{\textit{Imaginary Part of the } W - P \textit{ metric}\}$$

on $T_{\tau,\mathcal{M}(M)}$.

Proof: We proved in [32] that the natural metric

$$\|\Omega_\tau\|^2 = -\sqrt{-1}\int\limits_{\mathrm{M}} \Omega_\tau \wedge \overline{\Omega_\tau}$$

on $\omega_{\mathcal{Y}\ (\mathrm{M})\ /\mathcal{M}(\mathrm{M})}$ is such that its Chern form is

$$\sqrt{-1}\frac{\partial^2}{\partial\tau^i\partial\overline{\tau^j}}\log(\|\Omega_\tau\|^2) = -\omega(\tau).$$

This shows that $\mathcal{L} = \pi^*(\omega^*_{\mathcal{Y}(\mathrm{M})\ /\mathcal{M}(\mathrm{M})})$ is the prequantum line bundle on the tangent bundle $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$. This proves Theorem 60. ■

Corollary 61 *The prequantum line bundles on the bundle* $R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}$ *is* $p^*\left(\pi_*\left(\omega^*_{\mathcal{Y}(M)\ /\mathcal{M}(M)}\right)\right)$, *where*

$$p : R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)} \to \mathcal{M}\left(M\right), \omega_{\mathcal{Y}(M)/\mathcal{M}(M)}$$

is the relative dualizing line bundle and the metric on its fibre is defined by the L^2 *norm of the holomorphic three form. The Chern class of* $\pi^*\left(\omega^*_{\mathcal{Y}(M)/\mathcal{M}(M)}\right)$ *is given by the restriction of* $\omega_1(\tau)$ *on* $W_\tau = H^1\left(M,T^{1,0}_M\right)$.

5.2 The Geometric Quantization of $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathbf{M})/\mathcal{M}(\mathbf{M})}$

We will compute the projective flat connection on the Hilbert space bundle $\mathcal{H}^Q$ over the vector bundle $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. The prequantum line bundle is the line bundle $p^*\left(\omega^*_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}\right)$ over $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})\mathcal{M}(\mathrm{M})}$.

Theorem 62 *The matrix of the operator* (dJ) *in the basis*

$$\left\{\frac{\partial}{\partial\tau^i}\Omega_\tau,\ i = 1, ..., N\right\}$$

is given on each fibre of $R^1\pi_*\Omega^2_{\mathcal{Y}(M)/\mathcal{M}(M)}$ *by,*

$$(dJ)^{\overline{b}}_{\underline{a}} = \sum_{c,d} C_{\underline{acd}}\, g^{d,\overline{b}}. \tag{76}$$

Proof: The proof of Theorem 62 follows directly from the following Lemma:

Lemma 63 *We have*

$$\left(\frac{\partial}{\partial\tau^i}J\right)\left(\frac{\partial}{\partial\tau^j}\Omega_\tau\right) = \sum_{k,l=1}^{N} C_{ijk}\, g^{k,\overline{l}}\,\overline{\frac{\partial}{\partial\tau^l}\Omega_\tau}, \tag{77}$$

where

$$C_{ijk} = -\sqrt{-1}\int_M \left(\frac{\partial^2}{\partial\tau^i\partial\tau^j}\Omega_\tau\right)\wedge\left(\frac{\partial}{\partial\tau^k}\Omega_\tau\right)|_{\tau=0} =$$

$$\sqrt{-1}\int_M \left(\Omega_\tau\right)\wedge\left(\frac{\partial^3}{\partial\tau^i\partial\tau^j\partial\tau^k}\Omega_\tau\right)|_{\tau=0}\ . \tag{78}$$

The idea of the proof of formula (77) is the following one; we know that in the basis

$$\left\{\frac{\partial}{\partial\tau^i}\Omega_\tau,\ i = 1, ..., N\right\}$$

of $H^1(\mathrm{M}_\tau, \Omega^2_\tau)$ the complex structure operator is given by the matrix:

$$\begin{pmatrix} \sqrt{-1}I_{h^{1,2}} & 0 \\ 0 & -\sqrt{-1}I_{h^{1,2}} \end{pmatrix}. \tag{79}$$

Thus (79) implies

$$\left(\frac{\partial}{\partial\tau^i}J\right)\left(\frac{\partial}{\partial\tau^j}\Omega_\tau\right)|_{\tau=0} = J\left(\frac{\partial^2\Omega_\tau}{\partial\tau^i\partial\tau^j}\right)|_{\tau=0}.$$

So we need to compute the expression of the vectors $\left\{\frac{\partial^2}{\partial\tau^i\partial\tau^j}\Omega_\tau,\ i,j=1,...,N\right\}$ as a linear combinations of

$$\left\{\overline{\frac{\partial}{\partial\tau^i}\Omega_\tau},\ i=1,...,N\right\}. \tag{80}$$

We know from [32] that $\left\{\frac{\partial^2\Omega_\tau}{\partial\tau^i\partial\tau^j}\right\} \in H^2(\mathrm{M})$ where $\tau = (\tau^1,...,\tau^N)$ are the flat coordinates. Therefore if we express the vectors

$$\left\{\frac{\partial^2\Omega_\tau}{\partial\tau^i\partial\tau^j}\right\} \in H^2\left(\mathrm{M}_\tau, \Omega^1_{\mathrm{M}_\tau}\right)$$

as linear combination of the basis (80), we will get explicitly the matrices of the operators $\frac{\partial}{\partial\tau^i}J_{\tau=0}$, $i=0,...,N$. From the explicit formulas of the operators $\frac{\partial}{\partial\tau^i}J|_{\tau=0}$ we will get the formula (77).

Proof: We know that

$$-\sqrt{-1}\int\limits_{\mathrm{M}}\left(\frac{\partial}{\partial\tau^i}\Omega_\tau\right)\wedge\left(\overline{\frac{\partial}{\partial\tau^j}\Omega_\tau}\right)|_{\tau=0} = \delta_{i,\overline{j}}. \tag{81}$$

By using the expression (6) for Ω_τ and the natural L^2 metric on $H^2(\mathrm{M})$ we get the following expression of the vector $\left(\frac{\partial^2}{\partial\tau^i\partial\tau^j}\Omega_\tau\right)|_{\tau=0}$ in the orthogonal basis

$$\left\{\overline{\frac{\partial}{\partial\tau^k}\Omega_\tau}|_{\tau=0},\ k=1,...,N\right\}$$

of $H^{2,1}(\mathrm{M})$:

$$\left(\frac{\partial^2}{\partial\tau^i\partial\tau^j}\Omega_\tau\right)|_{\tau=0} = -\sqrt{-1}\sum_{k=1}^N\left(\left\langle\frac{\partial^2}{\partial\tau^i\partial\tau^j}\Omega_\tau, \overline{\frac{\partial}{\partial\tau^k}\Omega_\tau}\right\rangle\right)\left(\overline{\frac{\partial}{\partial\tau^k}\Omega_\tau}\right)|_{\tau=0} =$$

$$-\sqrt{-1}\sum_{k=1}^N\left(\int\limits_{\mathrm{M}}\left(\frac{\partial^2}{\partial\tau^i\partial\tau^j}\Omega_\tau\right)\wedge\left(\frac{\partial}{\partial\tau^k}\Omega_\tau\right)\right)\left(\overline{\frac{\partial}{\partial\tau^k}\Omega_\tau}\right)|_{\tau=0} =$$

$$- \sqrt{-1}\sum_{k=1}^{N} C_{ijk} g^{k,\bar{l}} \left(\overline{\frac{\partial}{\partial \tau^{l}}\Omega_{\tau}}\right) |_{\tau=0} \tag{82}$$

Formula (82) implies that for any $\tau \in \mathcal{K}$ we have

$$\frac{\partial^2}{\partial \tau^i \partial \tau^j}\Omega_\tau = -\sqrt{-1}\sum_{k=1}^{N} C_{ijk} g^{k,\bar{l}} \left(\overline{\frac{\partial}{\partial \tau^{l}}\Omega_{\tau}}\right). \tag{83}$$

We know that

$$J\left(\frac{\partial}{\partial \tau^i}\Omega_\tau\right) = \sqrt{-1}\left(\frac{\partial}{\partial \tau^i}\Omega_\tau\right). \tag{84}$$

Combining (83) and (84) we get that

$$\frac{\partial}{\partial \tau^j}\left(J\left(\frac{\partial}{\partial \tau^i}\Omega_\tau\right)\right) = \frac{\partial}{\partial \tau^j}\left(\sqrt{-1}\left(\frac{\partial}{\partial z^i}\Omega_\tau\right)\right) =$$

$$\sqrt{-1}\left(\frac{\partial^2}{\partial \tau^j \partial z^i}\Omega_\tau\right) = \sum_{k=1}^{N} C_{ijk} g^{k,\bar{l}} \left(\overline{\frac{\partial}{\partial \tau^{l}}\Omega_{\tau}}\right). \tag{85}$$

Lemma 63 is proved. ■

Lemma 63 implies directly Theorem 62. ■

Corollary 64 *The projective flat connection on the Hilbert space bundle $\mathcal{H}^Q$ over the vector bundle $R^1\pi_*\Omega^2_{\mathcal{X}/\mathcal{K}}$ is given by*

$$\frac{\partial}{\partial \tau^a} + \frac{\sqrt{-1}}{4}\sum \overline{C_{abc}} g^{\bar{b},i} g^{\bar{c},j} \frac{D}{Dt^i}\frac{D}{Dt^j}. \tag{86}$$

Proof: According to Theorem 62 on $R^1\pi_*\Omega^2_{\mathcal{X}/\mathcal{K}}$ we have

$$\left(dJ\omega_1^{-1}\right)^{ij} D_i D_j = -\sqrt{-1}\sum_{a=1}^{N} \overline{C_{abc}} g^{\bar{b},i} g^{\bar{c},j} D_i D_j. \tag{87}$$

Thus the projective connection on $R^1\pi_*\Omega^2_{\mathcal{X}/\mathcal{K}}$ is given by (86). ■

5.3 Computations on the Tangent Bundle of the Moduli Space

To quantize geometrically the tangent bundle $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ on the moduli space $\mathcal{M}(\mathrm{M})$ means to compute explicitly the prequantum line bundle and then projective connection on the Hilbert space bundle $\mathcal{H}^Q$ related to the prequantum line bundle on the tangent bundle $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ of the moduli space $\mathcal{M}(\mathrm{M})$.

Theorem 65 *We have on $\mathcal{T}_{\mathcal{M}(M)}$*

$$(dJ)_i^{\overline{l}} = e^K \sum_{j,k,l=1}^{N} C_{ijk} G^{k,\overline{l}}. \tag{88}$$

Proof: Formula (88) follows directly from formula (87), the globally defined isomorphism $\iota_\tau : H^1\left(\mathrm{M}, T_{\mathrm{M}}^{1,0}\right) \approxeq H^1\left(\mathrm{M}, \Omega_{\mathrm{M}}^2\right)$, given by $\phi \to \phi \lrcorner(\eta_\tau)$, and the relation $g_{i,\overline{j}} = e^{-K} G_{i,\overline{j}}$, where e^{-K} is given by (73) . Theorem 65 is proved. ■

5.4 BCOV Anomaly Equations

Definition 66 *The following data,* ***a.*** *The moduli space $\mathcal{M}(M)$ of CY threefolds,* ***b.*** *The CHSV $\mathbb{S}p(2h^{2,1}, \mathbb{R})$ flat connection on the tangent bundle constructed in ,* ***c.*** *The "prequantized line bundle" $\pi^*(\omega^*_{\mathcal{Y}(M)/\mathcal{M}(M)})$ on the tangent bundle $\mathcal{T}_{\mathcal{M}(M)}$,* ***d.*** *The imaginary part of the Weil-Petersson metric ω and* ***e.*** *the bundle of the Hilbert spaces $\mathcal{H}^Q$ over the tangent bundle $\mathcal{T}_{\mathcal{M}(M)}$ with a projective flat connection on it, will be called a CY quantum system.*

Theorem 67 *The expression of Witten projective flat connection as defined in Definition 53 in the flat coordinates for the CY quantum system defined in Definition 66 coincides with the BCOV anomaly equations (1) in [4].*

Proof: In order to prove Theorem 67 we need to compute explicitly the Witten projective connection constructed in [2] on the Hilbert vector bundle $\mathcal{H}^Q$ over the tangent bundle of the moduli space $\mathcal{M}(\mathrm{M})$. The explicit formula (68) and since BCOV anomaly equations were established in the flat coordinate system $(\tau^1, ..., \tau^N)$ imply that we need to compute the expression of

$$(dJ\omega^{-1})^{ij} D_i D_j$$

on $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ in the same coordinate system. As we pointed out the flat coordinate system $(\tau^1, .., \tau^N) \in \mathcal{K}$ introduced on the basis of Theorem 6 is same coordinate system used in [4].

We already established in Theorem 62 the explicit expression of $(dJ\omega_1^{-1})^{ij} D_i D_j$ on $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ in the coordinates $(\tau^1, ..., \tau^N)$. We will establish first the local expression of $(dJ\omega^{-1})^{ij} D_i D_j$ on $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ in the coordinate system $(t^1, ..., t^N)$ defined by the identification $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ with $\omega^*_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ that uses the parallel section η_τ^{-1} of $\omega^*_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. Then we will compute $(dJ\omega^{-1})^{ij} D_i D_j$ on $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ in the flat coordinate $(\tau^1, ..., \tau^N)$ defined by the identification of $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ with $\omega^*_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ by the holomorphic tensor Ω_τ^{-1}. Thus we will establish BCOV anomaly equations (1) in [4] as the projective Witten connection. Theorem 67 will be proved.

In order to compute $\left(dJ\omega^{-1}_{1,\tau} D_i D_j\right)$ we need to established the relations between $g^{\overline{i},j}$ and $G^{\overline{i},j}$. (73) implies that these relations given by

$$g^{\overline{i},j} = e^K G^{\overline{i},j}, \tag{89}$$

where $e^{-K} = \|\eta_\tau\|^2$.

Let $(t^1, ..., t^N)$ be the complex linear coordinates on $T_{\tau,\mathcal{M}(\mathrm{M})}$ defined by the identification of $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ with $\omega^*_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})} \otimes R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ that uses that parallel section η of $\omega^*_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. Remark 59 and the relation $\Omega_\tau = \lambda(\tau)\eta_\tau$ imply that we have

$$(\tau^1, .., \tau^N) = \lambda(t^1, .., t^N). \tag{90}$$

We know that the expression of $\left(dJ\omega_1^{-1}\right)^{ij} D_i D_j$ on the fibre of $R^1\pi_*\Omega^2_{\mathcal{X}/\mathcal{K}}$ is given by (86) in the coordinates $(\tau^1, .., \tau^N)$. Thus combining (89) with (86) we get that on $T_{\tau,\mathcal{M}(\mathrm{M})}$ in the coordinates $(t^1, .., t^N)$ we have:

$$\left(dJ\omega^{-1}\right)^{ij} D_i D_j = -\sqrt{-1}e^{2K} \sum_{a=1}^{N} \overline{C_{abc}} G^{\overline{b},i} G^{\overline{c},j} D_i D_j. \tag{91}$$

From (91) and (68) we get that the projective connection on the tangent bundle of $\mathcal{M}\,(\mathrm{M})$ is given by

$$\frac{\partial}{\partial \tau^a} + \frac{\sqrt{-1}}{4} e^{2K} \sum \overline{C_{abc}} G^{\overline{b},i} G^{\overline{c},j} \frac{D}{Dt^i} \frac{D}{Dt^j} \tag{92}$$

By using the symplectic identifications of the fibres of $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ by using the flat $\mathbb{S}p(2h^{2,1}, \mathbb{R})$ and then by using the projective flat connection on the Hilbert space fibration $\mathcal{H}^Q$ we obtain that the quantum state represented by a parallel vector $\Psi(\tau, t)$ is independent of t. This means that (92) implies that if $\Psi(\tau, t)$ is independent of t then $\Psi(\tau, t)$ satisfies the following equations:

$$\left(\frac{\partial}{\partial \tau^a} + \frac{\sqrt{-1}}{4} e^{2K} \sum \overline{C_{abc}} G^{\overline{b},i} G^{\overline{c},j} \frac{D}{Dt^i} \frac{D}{Dt^j}\right) \Psi(\tau, t) = 0 \tag{93}$$

and

$$\frac{\partial}{\partial \overline{t^k}} \Psi(\tau, t) = 0. \tag{94}$$

Based on (90) we have

$$\frac{D}{Dt^i} = \lambda \frac{D}{D\tau^i}.$$

Thus formulas (93) and (94) can be written as follows:

$$\left(\frac{\partial}{\partial \tau^a} + \frac{\sqrt{-1}}{4} \lambda^2 e^{2K} \sum \overline{C_{abc}} G^{\overline{b},i} G^{\overline{c},j} \frac{D}{D\tau^i} \frac{D}{D\tau^j}\right) \Psi(\tau) = 0 \tag{95}$$

and

$$\frac{\partial}{\partial \overline{\tau^k}} \Psi(\tau) = 0. \tag{96}$$

As we pointed out the flat coordinate system $(\tau^1, .., \tau^N) \in \mathcal{K}$ introduced on the basis of Theorem 6 is same coordinates are use in [4]. Thus the BCOV anomaly equations (1) in [4] are the same as the equations (95) and (96).

Direct computations show that if F_g satisfy the equations (1) then

$$\Psi(\tau) = \exp\left(\sum_{g=1}^{\infty} \lambda^{2g-2} F_g\right)$$

satisfy (95). Theorem 67 is proved. ■

5.5 Comments

- Bershadsky, Cecotti, Ooguri and Vafa used for the free energy $Z(\lambda; t, \overline{t})$ the following expression:

$$\mathfrak{F}(\lambda; t, \overline{t}) := \sum_{g=1}^{\infty} \lambda^{2g-2} F_g \text{ and } Z(\lambda; t, \overline{t}) = \exp\left(\mathfrak{F}(\lambda; t, \overline{t})\right). \qquad (97)$$

  Compare this with the expression (2) for Z, i.e. $Z = \exp\left(\frac{1}{2}\mathfrak{F}(\lambda; t, \overline{t})\right)$ used by Witten in [37].

- It is proved in [4] by using physical arguments that g-genus partition function F_g satisfy the equation (1). The holomorphic anomaly equation (**3.8**) derived in [4] is

$$\left(\overline{\frac{\partial}{\partial t^i}} - \overline{\frac{\partial}{\partial t^i}} F_1\right) \exp \mathfrak{F}(\lambda; t, \overline{t}) = \frac{\lambda^2}{2} \overline{C}_{ijk} e^{2K} G^{i,\overline{j}} G^{k,\overline{k}} \hat{D}_j \hat{D}_k \exp \mathfrak{F}(\lambda; t, \overline{t}), \qquad (98)$$

  where

$$\hat{D}_j \mathfrak{F}(\lambda; t, \overline{t}) = \sum_g \lambda^{2g-2} D_j F_g =$$

$$\sum_g \lambda^{2g-2} \left(\partial_j - (2g-2)\, \partial_j K\right) F_g =$$

$$\left(\partial_j - \partial_j K \lambda \partial_\lambda\right) \mathfrak{F}(\lambda; t, \overline{t}).$$

  Thus the equation (98) is different by term involving F_1 from the equation (95).

- In [4] the authors used the normalized holomorphic form, namely they normalized Ω_τ in such a way that $\int_\gamma \Omega_\tau = 1$, where γ is the invariant vanishing cycle. This normalized form is the same as the form defined in 9. They showed by using string theory that the functions F_g "count" curves of genus g. It seems to me that it is a very deep mathematical fact.

6 The Extended Period Space of CY Manifolds.

6.1 Definition of the Extended Period Space and Basic Properties

In this paragraph we will study the extended period space $\mathfrak{h}_{2,2h^{2,1}} \subset \mathbb{P}(H^3(\mathrm{M},\mathbb{C})$, which parametrizes all possible filtrations of the type:

$$F^0 = H^{3,0} \subset F^1 = H^{3,0} + H^{2,1} + H^{1,2} \subset F^2 = H^3(\mathrm{M},\mathbf{C}),$$

where $\dim F^0 = 1$ plus some extra properties which are motivated from the above filtration and Variations of Hodge Structures on K3 surfaces.

We will use the following notation for the cup product for Ω_1 and $\Omega_2 \in H^3(\mathrm{M},\mathbb{C})$, i.e. $\langle \Omega_1, \Omega_2 \rangle = \int\limits_{\mathrm{M}} \Omega_1 \wedge \Omega_2$.

Definition 68 $\mathfrak{h}_{2,2h^{2,1}}$ *by definition is the set of lines in $H^3(M,\mathbb{C})$ spanned by the cohomology class $[\Omega]$ of the holomorphic three form in $H^3(M,\mathbb{C})$ such that*

$$-\sqrt{-1}\int\limits_{M} \Omega \wedge \overline{\Omega} = -\sqrt{-1}\left\langle \Omega, \overline{\Omega} \right\rangle > 0. \tag{99}$$

Theorem 69 *There is a one to one map between the points τ of $\mathfrak{h}_{2,2h^{1,2}}$ and the two dimensional oriented planes E_τ in $H^3(M,\mathbb{R})$, where E_τ is generated by γ_1 and μ_1 such that $\langle \gamma_1, \mu_1 \rangle = 1$.*

Proof: Let $[\Omega_\tau] \in H^3(\mathrm{M},\mathbb{C})$ be a non-zero vector corresponding to a point $\tau \in \mathfrak{h}_{2,2h^{1,2}}$. Since the class of cohomology $[\Omega_\tau]$ satisfies (99) we may chose Ω_τ such that

$$-\sqrt{-1}\left\langle \Omega_\tau, \overline{\Omega_\tau} \right\rangle = 2. \tag{100}$$

Then (99) and (100) imply $\int\limits_{\mathrm{M}} \operatorname{Im}\Omega_\tau \wedge \operatorname{Re}\Omega_\tau = \langle \operatorname{Re}\Omega_\tau, \operatorname{Im}\Omega_\tau \rangle = 1$. We will define E_τ to be the the two dimensional oriented subspace in $H^3(\mathrm{M},\mathbb{R})$ spanned by $\operatorname{Re}\Omega_\tau$ and $\operatorname{Im}\Omega_\tau$. So to each point $\tau \in \mathfrak{h}_{2,2h^{1,2}}$ we have assigned an oriented two plane E_τ in $H^3(\mathrm{M},\mathbb{R})$.

Suppose that E is a two dimensional oriented plane in $H^3(\mathrm{M},\mathbb{R})$ spanned by γ and $\mu \in H^3(\mathrm{M},\mathbb{R})$ be such that $\langle \gamma, \mu \rangle = 1$. Let $\Omega_E : \mu + \sqrt{-1}\gamma$. Then we have

$$-\sqrt{-1}\int\limits_{\mathrm{M}} \Omega_E \wedge \overline{\Omega_E} = -\sqrt{-1}\left\langle \Omega_E, \overline{\Omega_E} \right\rangle = 2\left\langle \gamma, \mu \right\rangle = 2.$$

So to the plane E we assign the line in $H^3(\mathrm{M},\mathbb{C})$ spanned by Ω_E. This proves Theorem 69. ■

Corollary 70 $\mathfrak{h}_{2,2h^{1,2}}$ *is an open set in* $\mathbf{Grass}(2, 2h^{1,2}+2)$. *So it has a complex dimension $2h^{1,2}$.*

Remark 71 *It is easy to see that each point* $\tau \in \mathfrak{h}_{2,2h^{1,2}}$ *defines a natural filtration of length two in* $H^3(M,\mathbb{C})$. *Indeed, let* $H^{3,0}_\tau$ *be the subspace in* $H^3(M,\mathbb{C})$ *spanned by a non-zero element* $\Omega_\tau \in F^0$. *Let* $\gamma_0 = \operatorname{Re}\Omega_\tau$ *and* $\operatorname{Im}\Omega_\tau = \mu_0$. *From Theorem 69 we know that* $\langle\gamma_0, \mu_0\rangle$ *is a positive number. Let*

$$\{\gamma_0, \mu_0, .., \gamma_{h^{1,2}}, \mu_{h^{1,2}}\}$$

be a symplectic basis of $H^3(M,\mathbb{R})$ *such that* $\langle\gamma_i, \mu_j\rangle = -\delta_{ij}$. *Let* $\Omega_i := \mu_i + \sqrt{-1}\gamma_i$. *We will define* $H^{2,1}_\tau$ *to be the subspace in* $H^3(M,\mathbb{C})$ *spanned by the vectors* Ω_i *for* $i = 1, .., h^{1,2}$. *Then we define* $H^{1,2}_\tau := \overline{H^{2,1}_\tau}$. *It is easy to see that*

$$\left(H^{2,1}_\tau + H^{1,2}_\tau\right)^\perp = H^{3,0}_\tau + H^{0,3}_\tau,$$

where $H^{0,3}_\tau := \overline{H^{3,0}_\tau}$ *and the orthogonality is with respect to*

$$\langle\omega_1, \overline{\omega_2}\rangle = \int_M \omega_1 \wedge \overline{\omega_2}.$$

The natural filtration in $H^3(M,\mathbb{R})$ *that corresponds to* $\tau \in \mathfrak{h}_{2,2h^{1,2}}$ *is defined as follows:*

$$F^0_\tau = H^{3,0}_\tau \subset F^1_\tau = H^{3,0}_\tau + H^{2,1}_\tau + H^{1,2}_\tau \subset H^3(M, \mathbb{C}). \tag{101}$$

Next we will introduce a metric G on H^3(M,$\mathbf{C}$). We will use the metric G to show that the filtration defined by 101 is a Hodge filtration of weight two.

Definition 72 *Let M be a fixed CY manifold and let*

$$\Omega = \Omega^{3,0} + \Omega^{2,1} + \Omega^{1,2} + \Omega^{0,3} \in H^3(M, \mathbf{C})$$

be the Hodge decomposition of Ω, *then* $G(\Omega, \overline{\Omega})$ *is defined as follows:*

$$G(\Omega, \overline{\Omega}) := -\sqrt{-1}\left(\int_M \Omega^{3,0} \wedge \overline{\Omega^{3,0}} + \int_M \Omega^{2,1} \wedge \overline{\Omega^{2,1}}\right) +$$

$$-\sqrt{-1}\left(\int_M \overline{\Omega^{1,2}} \wedge \Omega^{1,2} + \int_M \overline{\Omega^{0,3}} \wedge \Omega^{0,3}\right). \tag{102}$$

From the definition of the metric, it follows that it has a signature $(2, 2h^{2,1})$ on $\mathrm{H}^3(\mathrm{M},\mathbb{R})$. We will denote the quadratic form of this metric by Q.

Lemma 73 *The metric defined by* (*102*) *does not depend on the choice of the complex structure on M.*

Proof: Let M_0 and M_τ be two different complex structures on M. Let Ω_0 and Ω_τ be two non zero holomorphic three forms on M_0 and M_τ respectively. Let $\{\Omega_{0,i}\}$ and $\{\Omega_{\tau,i}\}$ be two bases of $H^{2,1}(\mathrm{M}_0)$ and $H^{2,1}(\mathrm{M}_\tau)$ respectively, where $i = 1,..,h^{2,1}$ such that $\left\langle \Omega_{0,i},\overline{\Omega_{0,j}}\right\rangle = \left\langle \Omega_{\tau,i},\overline{\Omega_{\tau,j}}\right\rangle = \delta_{ij}$. Then one see immediately that

$$\{\operatorname{Re}\Omega_0, \operatorname{Im}\Omega_0, ..., \operatorname{Re}\Omega_{0,i}, \operatorname{Im}\Omega_{0,i}, ..\} \textit{ and } \{\operatorname{Re}\Omega_\tau, \operatorname{Im}\Omega_\tau, .., \operatorname{Re}\Omega_{\tau,i}, \operatorname{Im}\Omega_{\tau,i}, ..\}$$

are two different symplectic bases of $H^3(\mathrm{M},\mathbb{R})$. So there exists an element $\mathrm{g}\in \mathbb{S}p(2h^{2,1}+2)$ such that

$$\mathrm{g}(\Omega_0) = \Omega_\tau \text{ and } \mathrm{g}(\Omega_{0,i}) = \Omega_{\tau,i}. \tag{103}$$

So (103) implies that

$$\mathrm{g}(H^{2,1}(\mathrm{M}_0)) = H^{2,1}(\mathrm{M}_\tau) \tag{104}$$

and

$$\mathrm{g}(H^{3,0}(\mathrm{M}_0)) = H^{3,0}(\mathrm{M}_\tau) \tag{105}$$

Lemma 73 follows directly from (104), (105) and the definition of the metric G. ■

Theorem 74 *Let $\mathfrak{h}_{2,2h^{1,2}}$ be the space that parametrizes all filtrations in $H^3(M,\mathbb{C})$ defined in Remark 71. Then these filtration are Hodge filtrations of weight two and their moduli space is isomorphic to the symmetric space*

$$\mathfrak{h}_{2,2h^{1,2}} := \mathbb{SO}_0(2,2h^{1,2})/\mathbb{SO}(2)\times\mathbb{SO}(2h^{1,2}).$$

Proof: Since the signature of the metric G is $(2,2h^{2,1})$ the proof of Theorem 74 is standard and follows directly from definition of the variations of Hodge structures of weight two. See for example [15]. ■

We will use the fact that the space $\mathfrak{h}_{2,2h^{1,2}} = \mathbb{SO}_0(2,2h^{1,2})/\mathbb{SO}(2)\times\mathbb{SO}(2h^{1,2})$ is as an open set in the Grassmannian $Grass(2,b_3)$ identified with all two dimensional oriented subspaces in $H^3(\mathrm{M},\mathbb{R})$ such that the restriction of the quadratic form Q on them is positive, i.e.

$$\mathfrak{h}_{2,h^{1,2}} := \{E\subset H^3(\mathrm{M},\mathbb{R}) \dim E = 2, Q\mid_E > 0 + orientation\}.$$

Definition 75 *We will define a canonical map from $\mathfrak{h}_{2,2h^{1,2}}\subset Gr(2,2h^{1.2}+2)$ to $\mathbb{P}(H^3(M,\mathbb{R})\otimes\mathbb{C})$ as follows; let $E_\tau\in\mathfrak{h}_{2,2h^{1,2}}$, i.e. E_τ is an oriented two dimensional subspace in $H^3(M,\mathbb{R})$ on which the restriction of Q is positive. Let e_1 and e_2 be an orthonormal basis of E_τ. Let $\Omega_\tau :=\ e_1+\sqrt{-1}e_2$. Then Ω_τ defines a point $\tau\in\mathbb{P}(H^3(M,\mathbb{Z})\otimes\mathbb{C}))$ that corresponds to the line in $H^3(M,\mathbb{R})\otimes\mathbb{C})$ spanned by Ω_τ. It is a standard fact that the points τ in $\mathbb{P}(H^3(M,\mathbb{R})\otimes\mathbb{C}))$ is such that $Q(\tau,\tau)= 0$ and $Q(\tau,\overline{\tau})>0$ are in one to one corresponds with the points in $\mathfrak{h}_{2,h^{1,2}}$. See [31]. This follows from the arguments used in the proof of Theorem 69 or see [31].*

It is a well known fact that $\mathfrak{h}_{2,h^{1,2}}$ is isomorphic to one of the irreducible component of the open set of the quadric in $\mathbb{P}(H^3(\mathrm{M},\mathbb{R})\otimes\mathbb{C})$ defined as follows:

$$\mathfrak{h}_{2,2h^{1,2}} \approx \{\tau \in \mathbb{P}(H^3(\mathrm{M},\mathbb{Z})\otimes\mathbb{C})\,|Q(\tau,\tau)=0 \textit{ and } Q(\tau,\overline{\tau})>0\}.$$

(See [15].)

We will consider the family $\mathcal{X}\times\overline{\mathcal{X}} \to \mathcal{K}\times\overline{\mathcal{K}}$, where the family $\overline{\mathcal{X}} \to \overline{\mathcal{K}}$ is the family that corresponds to the conjugate complex structures, i.e. the point $(\tau_1,\overline{\tau_2}) \in \mathcal{K}\times\mathcal{K}$ τ_1 corresponds to the complex structure M_{τ_1} and $\overline{\tau_2}$ corresponds to the $\overline{\mathrm{M}_{\tau_2}}$, where $\overline{\mathrm{M}_{\tau_2}}$ is the conjugate complex structure on M_{τ_2}.

We will define the period map

$$p:\mathcal{K}\times\overline{\mathcal{K}}\to\mathbb{P}(H^3(\mathrm{M},\mathbb{Z})\otimes\mathbb{C}) \tag{106}$$

as follows; to each point $(\tau,\upsilon)\in\mathcal{K}\times\overline{\mathcal{K}}$ we will assign the complex line $p(\tau,\upsilon)$ in $H^3(\mathrm{M},\mathbb{R})\otimes\mathbb{C}$ defined by the oriented two plane $E_{\tau,\upsilon}\subset H^3(\mathrm{M},\mathbb{Z})\otimes\mathbb{R}$ spanned by $\mathrm{Re}(\Omega_\tau+\overline{\Omega_\upsilon})$ and $\mathrm{Im}(\Omega_\tau-\Omega_\upsilon)$, where Ω_τ and Ω_ν are defined as in Theorem 7. $\mathcal{K}$ is the Kuranishi space defined in Definition 10. We will show that the analogue of local Torelli Theorem holds, i.e. we will show that the period map p is a local embedding $p:\mathcal{K}\times\overline{\mathcal{K}}\subset\mathfrak{h}_{2,2h^{1,2}}$.

Remark 76 *We will define an embedding of the Kuranishi family $\mathcal{K}$ into $\mathcal{K}\times\overline{\mathcal{K}}$ as follows; to each $\tau\in\mathcal{K}$ we will associate the complex structure $(I_\tau,-I_\tau)$ on $M\times M$.*

Theorem 77 *The period map p defined by (106) is a local isomorphism. Moreover the image $p(\mathcal{K}\times\overline{\mathcal{K}})$ is contained in $\mathfrak{h}_{2,2h^{1,2}}\subset\mathbb{P}\left(H^3(M,\mathbb{Z})\otimes\mathbb{C}\right)$ for small enough ε.*

Proof: The fact that the period map p is a local isomorphism follows directly from the local Torelli theorem for CY manifolds proved in [14]. Let $(\tau,\overline{\tau})\in\triangle\subset\mathcal{K}\times\overline{\mathcal{K}}$, then clearly the point $p(\tau,\overline{\tau})\in\mathfrak{h}_{2,2h^{1,2}}\subset\mathbb{P}(H^3(\mathrm{M},\mathbb{R})\otimes\mathbb{C})$, i.e. $\mathrm{Q}|_{E_{\tau,\overline{\tau}}}>0$. This follows directly from the definition of Q and the fact that $E_{\tau,\overline{\tau}}$ is the subspace in $H^3(\mathrm{M}_\tau,\mathbb{R})$ spanned by $\mathrm{Re}\,\Omega_\tau$ and $\mathrm{Im}\,\Omega_\tau$.

Let $(\tau_1,\overline{\tau_2})\in\mathcal{K}\times\overline{\mathcal{K}}$ be a point "close" to the diagonal Δ in $\mathcal{K}\times\overline{\mathcal{K}}$ then $\mathrm{Q}\big|_{E_{\tau,\upsilon}}>0$. Indeed this follows from the fact that the condition $\mathrm{Q}\big|_{E_{\tau,\upsilon}}>0$ is an open one on $Gr(2,2h^{1,2}+2)$. Then the two dimensional oriented space $E_{(\tau_1,\overline{\tau_2})}\subset H^3(\mathrm{M},\mathbb{R})$ spanned by $\{\mathrm{Re}\,\Omega_{\tau_1}+\mathrm{Re}\,\Omega_{\tau_2},\mathrm{Im}\,\Omega_{\tau_1}+\mathrm{Im}\,\Omega_{\tau_2}\}$, where Ω_{τ_1} and Ω_{τ_2} are defined by formula (6) in Theorem 7 will be such that $\mathrm{Q}\big|_{E_{\tau,\upsilon}}>0$. From here we deduce that $p(\tau_1,\overline{\tau_2})\in\mathfrak{h}_{2,2h^{1,2}}$. Theorem 77 is proved. ■

We defined the Kuranishi space $\mathcal{K}$ to be a open polydisk $|\tau^i|<\varepsilon$ for $i=1,...,N$ in $H^1(\mathrm{M})$. Since p is a local isomorphism we may assume that $\mathcal{K}\times\overline{\mathcal{K}}$ is contained in $\mathfrak{h}_{2,2h^{1,2}}$ for small enouph ε.

6.2 Construction of a $\mathbb{Z}$ Structure on the Tangent Space of $\mathcal{M}(\mathrm{M})$

Definition 78 *To define a $\mathbb{Z}$ structure on a complex vector space V means the construction of a free abelian group $A\subset V$ such that the rank of A is equal to*

the dimension of V, i.e. $A \otimes \mathbb{C} = V$.

Definition 79 *We define* $\mathfrak{h}_{2,2h^{1,2}}(\mathbb{Q})$ *as follows; A point* $\tau \in \mathfrak{h}_{2,2h^{1,2}}(\mathbb{Q})$ *if the two dimensional oriented subspace* $E_\tau = H^{3,0}_\tau + H^{0,3}_\tau$ *that corresponds to* τ *constructed in Theorem 69 is such that* $E_\tau \subset H^3(M,\mathbb{Z}) \otimes \mathbb{Q}$.

Theorem 80 $\mathfrak{h}_{2,2h^{1,2}}(\mathbb{Q})$ *is an everywhere dense subset in* $\mathfrak{h}_{2,2h^{1,2}}$.

Proof: Our claim follows directly from two facts. The first one is that the set of the points in $Gr(2, 2 + 2h^{2,1})$ that corresponds to two dimensional subspaces in $H^3(\mathrm{M},\mathbb{Z}) \otimes \mathbb{Q}$ form an everywhere dense subset in $Gr(2, 2 + 2h^{2,1})$ and the second one is that $\mathfrak{h}_{2,2h^{2,1}}$ is an open set in $Gr(2, 2 + 2h^{2,1})$. Theorem 80 is proved. ■

Theorem 81 *For each* $\tau \in \mathfrak{h}_{2,2h^{2,1}}(\mathbb{Q})$ *a natural* $\mathbb{Z}$ *structure is defined on the tangent space* $T_{\tau,\mathfrak{h}_{2,2h^{2,1}}}$ *at the point* $\tau \in \mathfrak{h}_{2,2h^{2,1}}$. *This means that there exists a subspace* $\mathbb{Z}^{2h^{2,1}} \subset H^3(M,\mathbb{Z})$ *such that* $H^{2,1}_\tau + H^{1,2}_\tau \approxeq T_{\tau,\mathfrak{h}_{2,2h^{2,1}}} = \mathbb{Z}^{2h^{2,1}} \otimes \mathbb{R}$.

Proof: From the theory of Grassmannians we know that $T_{\tau,\mathfrak{h}_{2,2h^{2,1}}}$ can be identified with $H^{2,1}_\tau + H^{1,2}_\tau$. Our corollary follows directly from the construction of $H^{2,1}_\tau + H^{1,2}_\tau$ described in Remark 71. Indeed the point $\tau \in \mathfrak{h}_{2,2h^{2,1}}(\mathbb{Q})$ defines two vectors γ_0 and $\mu_0 \in H^3(\mathrm{M},\mathbb{Z})/Tor$ that form a basis of $H^{3,0}(\mathrm{M}) \oplus H^{0,3}(\mathrm{M})$ such that $\langle \mu_0, \gamma_0 \rangle \in \mathbb{Z}$ and $\langle \mu_0, \gamma_0 \rangle > 0$. We choose the vectors

$$\{\gamma_0, \mu_0, \gamma_1, \mu_1, .., \gamma_{h^{2,1}}, \mu_{h^{2,1}}\}$$

to be in $H^3(\mathrm{M},\mathbb{Z})$/Tor and we require that $\langle \mu_i, \gamma_j \rangle = \delta_{ij}$. Then, from the way we defined $H^{2,1}_\tau$ and $H^{1,2}_\tau$, it follows that

$$H^{2,1}_\tau + H^{1,2}_\tau = (\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\mu_1 \oplus ...\mathbb{Z}\gamma_{h^{2,1}} \oplus \mathbb{Z}\mu_{h^{2,1}}) \otimes \mathbb{R}.$$

Theorem 81 is proved. ■

We will consider the embedding of $\mathcal{K}$ in $\mathcal{K}\times\overline{\mathcal{K}}$ defined in Remark 76. Next we choose a point $\kappa \in \mathcal{K}\times\overline{\mathcal{K}}\subset\mathfrak{h}_{2,2h^{2,1}} \subset \mathbb{P}(H^3(\mathrm{M},\mathbb{Z}) \otimes \mathbb{C})$ such that $\kappa \in (\mathcal{K}\times\overline{\mathcal{K}}) \cap \mathfrak{h}_{2,2h^{2,1}}(\mathbb{Q})$. We know that κ corresponds to a two dimensional space $E_\kappa \subset H^3(\mathrm{M},\mathbb{Q})$, with the additional condition, that there exists vectors γ_0 and $\mu_0 \in H^3(\mathrm{M},\mathbb{Z})$/Tor that span E_κ and $\langle \mu_0, \gamma_0 \rangle = 1$. The existence of such points follows from the fact that the set of all two dimensional space, $E_\kappa \subset H^3(\mathrm{M},\mathbb{Q})$ such that there exists vectors γ_0 and $\mu_0 \in H^3(\mathrm{M},\mathbb{Z})/Tor$ and $\langle \mu_0, \gamma_0 \rangle = 1$ is an everywhere dense subset in $\mathfrak{h}_{2,2h^{2,1}}$. Let

$$\{\gamma_0, \mu_0, \gamma_1, \mu_1, .., \gamma_{h^{2,1}}, \mu_{h^{2,1}}\} \in H^3(\mathrm{M},\mathbb{Z})/Tor$$

be such that γ_0 and μ_0 span E_κ and $\langle \mu_i, \gamma_j \rangle = \delta_{ij}$. It follows from the construction in Remark 71 that the vectors

$$\gamma_1, \mu_1, .., \gamma_{h^{2,1}}, \mu_{h^{2,1}} \in H^3(\mathrm{M},\mathbb{Z})/Tor$$

span $H_{\kappa}^{2,1}+H_{\kappa}^{1,2}$, i.e. they span the tangent space $T_{\kappa,\mathfrak{h}_{2,2h^{2,1}}}=H_{\kappa}^{2,1}+H_{\kappa}^{1,2}$. We know from Corollary 47 that there exists an $\mathbb{S}p(2h^{2,1},\mathbb{R})$ flat connection on $\mathcal{K}$ and so we define a flat connection on the product $\mathcal{K}\times\overline{\mathcal{K}}$ as the direct sum of the two connections. Using this $\mathbb{S}p(4h^{2,1},\mathbb{R})$ flat connection we can perform a parallel transport of the vectors $\gamma_1,\mu_1,..,\gamma_{h^{2,1}},\mu_{h^{2,1}}\in T_\kappa$ to a basis

$$\gamma_{1,\tau},\mu_{1,\tau},..,\gamma_{h^{2,1},\tau},\mu_{h^{2,1},\tau}$$

in the tangent space $H_{\tau}^{2,1}+H_{\tau}^{1,2}\approxeq T_{\tau,\mathfrak{h}_{2,2h^{2,1}}}$ to each point $\tau\in\mathcal{K}\subset\mathcal{K}\times\overline{\mathcal{K}}$. Thus we can conclude that $\langle\mu_{i,\tau},\gamma_{j,\tau}\rangle=\delta_{ij}$ and the free abelian group

$$A_\tau:=\mathbb{Z}\gamma_{1,\tau}\oplus\mathbb{Z}\mu_{1,\tau}\oplus...\mathbb{Z}\gamma_{h^{2,1},\tau}\oplus\mathbb{Z}\mu_{h^{2,1},\tau}$$

in $H_{\tau}^{2,1}+H_{\tau}^{1,2}$ is such that

$$H_{\tau}^{2,1}+H_{\tau}^{1,2}=(\mathbb{Z}\gamma_{1,\tau}\oplus\mathbb{Z}\mu_{1,\tau}\oplus...\mathbb{Z}\gamma_{h^{2,1},\tau}\oplus\mathbb{Z}\mu_{h^{2,1},\tau})\otimes\mathbb{R}.$$

So we defined for each $\tau\in\mathcal{K}$ an abelian subgroup $A_\tau\subset\mathrm{T}_{\tau,\mathcal{K}}=\left(H_{\tau}^{2,1}+H_{\tau}^{1,2}\right)$ such that

$$A_\tau\otimes\mathbb{C}=T_{\tau,\mathcal{K}}=(H_{\tau}^{2,1}+H_{\tau}^{1,2})$$

and $\langle\gamma,\mu\rangle\in\mathbb{Z}$ for γ and $\mu\in A_\tau$.

Definition 82 *The image of the projection of the abelian subgroup A_τ of $T_{\tau,\mathcal{K}}=(H_{\tau}^{2,1}+H_{\tau}^{1,2})$ to $H_{\tau}^{2,1}$ will be denoted by Λ_τ for each $\tau\in\mathcal{M}(M)$.*

Theorem 83 *There exists a holomorphic map ϕ from the moduli space $\mathcal{M}(M)$ of CY manifolds to the moduli space of principally polarized abelian varieties $\mathbb{S}p(2h^{2,1},\mathbb{Z})\backslash\mathfrak{Z}_{h^{2,1}}$, where $\mathfrak{Z}_{h^{2,1}}:=\mathbb{S}p(2h^{2,1},\mathbb{R})/U(h^{2,1})$.*

Proof: From 82 we know that there exists a lattice $\Lambda_\tau\subset\mathrm{T}_{\tau,\mathcal{K}}$ such that the restriction of the imaginary part of the Weil-Petersson metric

$$\mathrm{Im}(g)(u,v)=\langle u,v\rangle$$

on Λ_τ is such that $\langle u,v\rangle\in\mathbb{Z}$ and $|\det\langle\gamma_i,\gamma_j\rangle|=1$ for any symplectic basis of Λ_τ. Thus over $\mathcal{K}$ we can construct a family of principally polarized abelian varieties

$$\mathcal{A}_K\to\mathcal{K}. \tag{107}$$

In fact we constructed a family of principally polarized abelian varieties

$$\mathcal{A}\to\mathcal{M}(\mathrm{M}) \tag{108}$$

over the moduli space $\mathcal{M}(\mathrm{M})$ since CHSV connection is a flat connection globally defined over $\mathcal{M}(\mathrm{M})$. This means that we defined the holomorphic map ϕ between the quasi-projective varieties

$$\phi:\mathcal{M}(\mathrm{M})\to\mathbb{S}p(2h^{2,1},\mathbb{Z})\backslash\mathfrak{Z}_{h^{2,1}}.$$

The existence of ϕ follows from the fact that there exists a versal family of principally polarized abelian varieties

$$\mathfrak{A} \to \mathbb{S}p(2h^{2,1},\mathbb{Z})\backslash\mathfrak{Z}_{h^{2,1}}.$$

Theorem 83 is proved. ■

Notice that the family of principally polarized varieties $\mathcal{A} \to \mathcal{M}(\mathrm{M})$ is constructed by using the vector bundle $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$. Using the identification between $\mathcal{T}_{\mathcal{M}(\mathrm{M})}$ and $R^1\pi_*\Omega^2_{\mathcal{Y}(\mathrm{M})/\mathcal{M}(\mathrm{M})}$ given by $\phi \to \phi\lrcorner\eta_\tau$, we define a family of principally polarized varieties isomorphic to the family $\mathcal{A} \to \mathcal{M}(\mathrm{M})$.

6.3 Holomorphic Symplectic Structure on the Extended Period Domain

Definition 84 *Let us fix a symplectic basis*

$$\{\gamma_0, \gamma_1, .., \gamma_{h^{1,2}}; \upsilon_0, ..., \upsilon_{h^{1,2}}\}$$

in $H^3(M,\mathbb{Z})/Tor$, i.e. $\langle\gamma_i, \upsilon_j\rangle = \delta_{ij}$. This basis defines a coordinate system in $\mathbb{P}(H^3(M,\mathbb{Z}) \otimes \mathbb{C})$ which we will denote by $(z_0 : .. : z_{2h^{1,2}})$. On the open set:

$$U_0 := \{(z_0 : .. : z_{2h^{1,2}+1})\,|z_0 \neq 0\},$$

we define a holomorphic one forms:

$$\alpha_0 := dt_{h^{1,2}+1} + t_1 dt_{h^{1,2}+2} + .. + t_{h^{1,2}} dt_{2h^{1,2}+1}, \tag{109}$$

where $t_i = \frac{z_i}{z_0}$, for $i = 1, .., h^{1,2}$. Let us restrict α_0 on $\mathfrak{h}_{2,2h^{1,2}} \cap U_0$ and denote this restriction by α_0. In the same way we can define the forms α_i on the open set $U_i := \{(z_0 : .. : z_{2h^{1,2}})|\ z_i \neq 0\}$.

Theorem 85 *There exists a closed holomorphic non degenerate two form ψ on $\mathfrak{h}_{2,2h^{1,2}}$ such that*

$$\psi\Big|_{U_i\cap\mathfrak{h}_{2,2h^{1,2}}} = d\alpha_i\Big|_{U_i\cap\mathfrak{h}_{2,2h^{1,2}}}. \tag{110}$$

Proof: The proof of Theorem 85 is based on the following Proposition:

Proposition 86 *We have*

$$d\alpha_i = d\alpha_j \tag{111}$$

on $\mathfrak{h}_{2,2h^{1,2}} \cap (U_i \cap U_j)$. Thus there exists a holomorphic non degenerate form ψ such that $\psi|_{U_i} = d\alpha_i$.

Proof: It is easy to see that since the extended period domain $\mathfrak{h}_{2,2h^{1,2}}$ is an open set of a quadric in $\mathbb{P}(H^3(\mathrm{M},\mathbb{Z}) \otimes \mathbb{C})$ then that the tangent space $T_{\tau,\mathfrak{h}_{2,2h^{1,2}}}$ to any point $\tau \in \mathfrak{h}_{2,2h^{1,2}}$ can be identified with the orthonormal complement $\left(H^{3,0}_\tau + \overline{H^{3,0}}\right)^\perp \subset H^3(\mathrm{M},\mathbb{C})$ with respect to the metric G defined in Definition

72. Let υ and $\mu \in T_{\tau,\mathfrak{h}_{2,2h^{1,2}}} \subset H^3(\mathrm{M},\mathbb{C})$. From the definition of the form $d\alpha_i$, it follows that

$$d\alpha_i(\upsilon,\mu) = \langle \upsilon,\mu \rangle\,, \tag{112}$$

where $\langle \upsilon,\mu \rangle$ is the symplectic form defined by the intersection form on $H^3(\mathrm{M},\mathbb{Z})$. From here Proposition 86 follows directly. ■

Theorem 85 follows directly from Proposition 86. ■

Corollary 87 *The holomorphic two form ψ is a parallel form when restricted to $\mathcal{K}\times\mathcal{K} \subset \mathfrak{h}_{2,2h^{1,2}}$ with respect to the CHSV connection. (See Definition 38.)*

Proof: The corollary follows directly from Definition 38 and the fact that the imaginary part of the Weil-Petersson metric when restricted to the tangent space $T_{\tau,\mathcal{K}} = H^{2,1} \subset H^3(\mathrm{M},\mathbb{C})$ is just the restriction of intersection form $\langle \upsilon,\mu \rangle$ on $H^3(\mathrm{M},\mathbb{C})$. ■

7 Algebraic Integrable System on the Moduli Space of CY Manifolds.

Definition 88 *Let N be an algebraic variety. An algebraic integrable system is a holomorphic map $\pi : X \to N$ where* ***a.*** *X is a complex symplectic manifold with holomorphic symplectic form $\psi \in \Omega^{2,0}(X)$;* ***b.*** *The fibres of π are compact Lagrangian submanifolds, hence affine tori;* ***c.*** *There is a family of smoothly varying cohomology classes $[\rho_n] \in H^{1,1}(X_n) \cap H^2(X_n,\mathbb{Z}), n \in N$, such that $[\rho_n]$ is a positive polarization of the fibre X_n. Hence X_n is an abelian tersor, i.e. on X_n we do not have a point which represents zero to define a structure of a group on X_n. See [9].*

This notion is the complex analogue of completely integrable (finite dimensional) systems in classical mechanics was introduced by R. Donagi and E. Markman in [9]. We will show that the family $\mathcal{A} \to \mathcal{M}(\mathrm{M})$ as defined in Definition 82 is an algebraic integrable system in the sense of Donagi-Markrman.

Theorem 89 *The holomorphic family $\mathcal{A} \to \mathcal{M}(M)$ defines an algebraic integrable system on the moduli space of three dimensional CY manifolds $\mathcal{M}(M)$ in the sense of R. Donagi and Markman.*

Proof: We must check properties **a**, **b** and **c** stated in **Definition** 88. In order to check property **a** and **b,** we need to construct a non-degenerate closed holomorphic two form Ω_1 on the cotangent space $T^*\mathcal{K}(\mathrm{M})$. The cotangent space $T^*_\tau(\mathrm{M})$ at a point $\tau \in \mathcal{K}(\mathrm{M})$ can be identified with $H^{1,2}(\mathrm{M}_\tau)$ by contraction with $\overline{\Omega_\tau}$, where Ω_τ is a holomorphic three form such that

$$-\sqrt{-1}\int_{\mathrm{M}} \Omega_\tau \wedge \overline{\Omega_\tau} = 1.$$

Then the local Torelli theorem for CY manifolds shows that the restriction of the symplectic form $\langle \upsilon, \mu \rangle$ defined by the intersection form on $H^3(\mathrm{M},\mathbb{Z})/\mathrm{Tor}$ by formulas (110) and (112) will give a globally defined holomorphic two form Ω_1 on the cotangent bundle of $\mathcal{K}(\mathrm{M})$. Then the properties **a** and **b** as stated in [9] are obvious.

Next we will construct the smoothly varying cohomology classes $[\rho_\tau]$ which fulfill property **c.** Let $\rho(1,1)$ be the imaginary form of the Weil-Petersson metric on $\mathcal{M}(\mathrm{M})$. Since we proved in [32] that the potential of the Weil-Petersson metric is defined from a metric on the relative dualizing line bundle of the family $\mathcal{X} \to \mathcal{M}(\mathrm{M})$, we deduce that $\rho(1,1)$ is a smoothly varying cohomology class of type $(1,1)$. From here we deduce that for each $\tau \in \mathcal{M}(\mathrm{M})$, $[\rho_\tau] \in H^{1,1}(\mathcal{A}_\tau, \mathbb{R})$ and $[\rho_\tau]$ varies smoothly. We need to show that $[\rho_\tau] \in H^2(\mathcal{A}_\tau, \mathbb{Z})$. This statement is equivalent to saying that if ν and μ are any two vectors in the lattice

$$\Lambda_\tau \subset H^{2,1}(\mathrm{M}_\tau) \oplus H^{1,2}(\mathrm{M}_\tau),$$

then $\rho_\tau(\nu,\mu) \in \mathbb{Z}$ and if $\gamma_1, .., \gamma_{2h^{1,2}}$ is a $\mathbb{Z}$-basis of the lattice Λ_τ, then $\det(\rho_\tau(\gamma_i,\gamma_j)) = 1$. We proved that the imaginary part of the Weil-Petersson metric is a parallel with respect to the Cecotti-Hitchin-Simpson-Vafa connection. (See Remark 48.) We used the Cecotti-Hitchin-Simpson-Vafa parallel transport to define the $\mathbb{Z}$ structure on

$$T_{\tau,\mathcal{K}\subset\mathcal{K}\times\mathcal{K}} = H^{2,1}(\mathrm{M}) + H^{1,2}(\mathrm{M}).$$

From here it follows that the number $[\rho_\tau](\nu,\mu)$ is equal to the cup product of the parallel transport of the vectors ν and μ at a point $(\tau,\upsilon) \in \mathfrak{h}_{2,2h^{1,2}}(\mathbb{Q})$, which is an integer. Exactly the same arguments show that

$$\det(\rho_\tau(\gamma_i,\gamma_j)) = 1.$$

So the family $\mathcal{A} \to \mathcal{M}(\mathrm{M})$ fulfills properties **b** and **c.** Our Theorem is proved. ■

Corollary 90 *On the tangent bundle of the moduli space of polarized CY threefolds there exists a canonical Hyper-Kähler metric.*

Proof: Cor. 90 follows directly from [12]. ■

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